Electromagnetic fields and waves
Maxwell’s rainbow
• Maxwell’s equations
• Plane waves
• Pulses and group velocity
• Polarization of light
• Transmission and reflection at an interface
The concept of fields was introduced to explain “action at distance.” The physical observables are forces.

Macroscopic Maxwell’s equations deal with fields that are local spatial averages over microscopic fields associated with discrete charges. Charge and current densities are considered as continuous functions of space.
The conservation of charge is implicitly contained in Maxwell’s equations.

\[ \nabla \cdot \vec{D}(\vec{r}, t) = \rho(\vec{r}, t) \]

\[ \nabla \cdot \vec{B}(\vec{r}, t) = 0 \]

\[ \nabla \times \vec{E}(\vec{r}, t) + \frac{\partial \vec{B}(\vec{r}, t)}{\partial t} = 0 \]

\[ \nabla \times \vec{H}(\vec{r}, t) - \frac{\partial \vec{D}(\vec{r}, t)}{\partial t} = \vec{j}(\vec{r}, t) \]

\[ \nabla \cdot \nabla \times \vec{H}(\vec{r}, t) - \frac{\partial \nabla \cdot \vec{D}(\vec{r}, t)}{\partial t} = \nabla \cdot \vec{j}(\vec{r}, t) \]

\[ \nabla \cdot \vec{j}(\vec{r}, t) = 0 \quad \text{(continuity equation)} \]
Maxwell’s equations are incomplete. The fields are connected to one another by constitutive relations (material equations) describing the electromagnetic response of media.

\[
\vec{\nabla} \cdot \vec{D}(\vec{r}, t) = \rho(\vec{r}, t)
\]

\[
\vec{\nabla} \cdot \vec{B}(\vec{r}, t) = 0
\]

\[
\vec{\nabla} \times \vec{E}(\vec{r}, t) + \frac{\partial \vec{B}(\vec{r}, t)}{\partial t} = 0
\]

\[
\vec{\nabla} \times \vec{H}(\vec{r}, t) - \frac{\partial \vec{D}(\vec{r}, t)}{\partial t} = \vec{j}(\vec{r}, t)
\]

Polarization – material dependent!

\[
\vec{D}(\vec{r}, t) = \varepsilon_0 \vec{E}(\vec{r}, t) + \vec{P}(\vec{r}, t)
\]

\[\varepsilon_0 = 8.854 \times 10^{-12} \text{ F/m} = \text{permittivity of vacuum}\]

Magnetization

\[
\vec{H}(\vec{r}, t) = \mu_0^{-1} \vec{B}(\vec{r}, t) - \vec{M}(\vec{r}, t)
\]

\[\mu_0 = 4\pi \times 10^{-7} \text{ H/m} = \text{permeability of vacuum}\]
Relation between D and E

Response function (tensor)

\[
\vec{P}(\vec{r}, t) = \varepsilon_0 \int_\infty^{-\infty} \int_\infty^{-\infty} \int_\infty^{-\infty} \vec{\chi}(\vec{r} - \vec{r}', t - t') \cdot \vec{E}(\vec{r}', t') \, dV' \, dt'
\]

\[
\vec{P}(\vec{r}, t) = \varepsilon_0 \vec{\chi} \vec{E}(\vec{r}, t)
\]

\[
\vec{D}(\vec{r}, t) = \varepsilon_0 \vec{E}(\vec{r}, t) + \vec{P}(\vec{r}, t) \neq \varepsilon_0 \varepsilon \vec{E}(\vec{r}, t)
\]

Assumptions:

- a linear medium (\(P\) is proportional to \(E\))
- an isotropic medium
- an instantaneous response (no temporal dispersion)
- a local response (no spatial dispersion)
**Response**

\[ \vec{P}(\vec{r}, t) = \varepsilon_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \vec{\chi}(\vec{r} - \vec{r}', t - t') \cdot \vec{E}(\vec{r}', t') \, dV' \, dt' \]

**Temporal dispersion:** \( P \) (or \( D \)) at time \( t \) depends on \( E \) at all times \( t' \) previous to \( t \) (non-instantaneous response). Temporal dispersion is widely encountered phenomenon and it is important to accurately take it into account.

**Spatial dispersion:** \( P \) (or \( D \)) at a point \([x,y,z]\) also depends on the values of the electric field at neighboring points \([x',y',z']\). A spatially dispersive medium is therefore also called a nonlocal medium. Nonlocal effects can be observed at interfaces between different media or in metallic objects with sizes comparable with the mean-free path of electrons. In most cases of interest the effect is very weak and we can safely ignore it.
\[ \vec{P}(\vec{r}, t) = \varepsilon_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{x}(\vec{r} - \vec{r}', t - t') \cdot \vec{E}(\vec{r}', t') \, dV' \, dt' \] 

convolution in \((\vec{r}, t)\) space

use the four-dimensional Fourier transform

\[ \vec{P}(\vec{k}, \omega) = \varepsilon_0 \hat{x}(\vec{k}, \omega) \cdot \vec{E}(\vec{k}, \omega) \] 

multiplication in \((\vec{k}, \omega)\) space

the spatial frequency of the field

the angular frequency of the field

\[ \frac{dk_x \, dk_y \, dk_z}{(2\pi)^3} \]

\[ \vec{F}(\vec{r}, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \vec{F}(\vec{k}, \omega) e^{i\vec{k} \cdot \vec{r} - i\omega t} \, dV_k \, d\omega \] 

(the inverse Fourier transform \(\equiv\) IFT)

\[ \vec{F}(\vec{k}, \omega) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \vec{F}(\vec{r}, t) e^{-i\vec{k} \cdot \vec{r} + i\omega t} \, dV \, dt \] 

(the Fourier transform \(\equiv\) FT)
Constitutive relations

\[ \vec{P}(\vec{r}, t) = \varepsilon_0 \int_{-\infty}^{\infty} \int \int \vec{\chi}(\vec{r} - \vec{r}', t - t') \cdot \vec{E}(\vec{r}', t') \, dV' \, dt' \]

convolution in \((\vec{r}, t)\) space

- use the four-dimensional Fourier transform

\[ \vec{P}(\vec{k}, \omega) = \varepsilon_0 \vec{\chi}(\vec{k}, \omega) \cdot \vec{E}(\vec{k}, \omega) \]

multiplication in \((\vec{k}, \omega)\) space

- the spatial frequency of the field
- the angular frequency of the field

- non-instantaneous response leads to \(\omega\)-dependence
- non-local response leads to \(k\)-dependence
Constitutive relations

\[ \vec{P}(\vec{r}, t) = \varepsilon_0 \int_{-\infty}^{\infty} \int V \int \nabla (\vec{r} - \vec{r}', t - t') \cdot \vec{E}(\vec{r}', t') \, dV' \, dt' \quad \text{convolution in } (\vec{r}, t) \text{ space} \]

use the four-dimensional Fourier transform

\[ \vec{P}(\vec{k}, \omega) = \varepsilon_0 \vec{\chi}(\vec{k}, \omega) \cdot \vec{E}(\vec{k}, \omega) \quad \text{multiplication in } (\vec{k}, \omega) \text{ space} \]

\[ \vec{D}(\vec{r}, t) = \varepsilon_0 \vec{E}(\vec{r}, t) + \vec{P}(\vec{r}, t) \]
\[ \vec{D}(\vec{k}, \omega) = \varepsilon_0 \vec{E}(\vec{k}, \omega) + \vec{P}(\vec{k}, \omega) = \varepsilon_0 \left[ \vec{I} + \vec{\chi}(\vec{k}, \omega) \right] \cdot \vec{E}(\vec{k}, \omega) \]

\[ \vec{D}(\vec{k}, \omega) = \varepsilon_0 \varepsilon(\vec{k}, \omega) \cdot \vec{E}(\vec{k}, \omega) \]
\[ \vec{B}(\vec{k}, \omega) = \mu_0 \mu(\vec{k}, \omega) \cdot \vec{H}(\vec{k}, \omega) \]
\[ \vec{j}(\vec{k}, \omega) = \sigma(\vec{k}, \omega) \cdot \vec{E}(\vec{k}, \omega) \]

\[ \text{obtained in a similar fashion} \]

- We will assume isotropic materials and ignore spatial dispersion (k-dependence).
- We will discuss the frequency dependence later on.
Solution in the frequency domain

\[ \nabla \cdot \vec{D}(\vec{r}, t) = \rho(\vec{r}, t) \]

\[ \nabla \cdot \vec{B}(\vec{r}, t) = 0 \]

\[ \nabla \times \vec{E}(\vec{r}, t) + \frac{\partial \vec{B}(\vec{r}, t)}{\partial t} = 0 \]

\[ \nabla \times \vec{H}(\vec{r}, t) - \frac{\partial \vec{D}(\vec{r}, t)}{\partial t} = \vec{j}(\vec{r}, t) \]

We assume a monochromatic field

i.e.

\[ \vec{E}(\vec{r}, t) = \text{Re} \left[ \vec{E}(\vec{r}) e^{-i\omega t} \right] \]

\[ \vec{H}(\vec{r}, t) = \text{Re} \left[ \vec{H}(\vec{r}) e^{-i\omega t} \right] \]

Complex amplitudes they depend on the angular frequency \( \omega \) (the dependence is not explicitly shown)
Solution in the frequency domain

\[ \hat{\nabla} \cdot \vec{D}(\vec{r}, t) = \rho(\vec{r}, t) \]
\[ \hat{\nabla} \cdot \vec{B}(\vec{r}, t) = 0 \]
\[ \hat{\nabla} \times \vec{E}(\vec{r}, t) + \frac{\partial \vec{B}(\vec{r}, t)}{\partial t} = 0 \]
\[ \hat{\nabla} \times \vec{H}(\vec{r}, t) - \frac{\partial \vec{D}(\vec{r}, t)}{\partial t} = \vec{j}(\vec{r}, t) \]

\[ e^{-i\omega t} \]

\[ \hat{\nabla} \cdot \vec{D}(\vec{r}) = \rho(\vec{r}) \]
\[ \hat{\nabla} \cdot \vec{B}(\vec{r}) = 0 \]
\[ \hat{\nabla} \times \vec{E}(\vec{r}) - i\omega \vec{B}(\vec{r}) = 0 \]
\[ \hat{\nabla} \times \vec{H}(\vec{r}) + i\omega \vec{D}(\vec{r}) = \vec{j}(\vec{r}) \]

The same Eqs. are obtained for the spectral components
\[ \omega \] not shown

The spectrum (complex)
\[ \vec{E}(\vec{r}, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \vec{E}(\vec{r}, t) e^{i\omega t} dt \] (FT)
\[ \vec{E}(\vec{r}, \omega) = \vec{E}^*(\vec{r}, -\omega) \]

(a consequence for the spectral components)
Solution in the frequency domain

\[ \vec{\nabla} \cdot \vec{D}(\vec{r}, t) = \rho(\vec{r}, t) \]
\[ \vec{\nabla} \cdot \vec{B}(\vec{r}, t) = 0 \]
\[ \vec{\nabla} \times \vec{E}(\vec{r}, t) + \frac{\partial \vec{B}(\vec{r}, t)}{\partial t} = 0 \]
\[ \vec{\nabla} \times \vec{H}(\vec{r}, t) - \frac{\partial \vec{D}(\vec{r}, t)}{\partial t} = \vec{j}(\vec{r}, t) \]

\[ e^{-i \omega t} \]

Time domain

\[ \vec{\nabla} \cdot \vec{D}(\vec{r}) = \rho(\vec{r}) \]
\[ \vec{\nabla} \cdot \vec{B}(\vec{r}) = 0 \]
\[ \vec{\nabla} \times \vec{E}(\vec{r}) - i \omega \vec{B}(\vec{r}) = 0 \]
\[ \vec{\nabla} \times \vec{H}(\vec{r}) + i \omega \vec{D}(\vec{r}) = \vec{j}(\vec{r}) \]

Frequency domain

two equally valid conventions for expressing the time dependence

\[ e^{-i \omega t} \quad e^{i \omega t} \]

(any field with – convention) = (the same field with + convention)∗

(the complex conjugate of the field)
Solution in the frequency domain

\[ \nabla \cdot \vec{D} (\vec{r}, t) = \rho (\vec{r}, t) \]
\[ \nabla \cdot \vec{B} (\vec{r}, t) = 0 \]
\[ \nabla \times \vec{E} (\vec{r}, t) + \frac{\partial \vec{B} (\vec{r}, t)}{\partial t} = 0 \]
\[ \nabla \times \vec{H} (\vec{r}, t) - \frac{\partial \vec{D} (\vec{r}, t)}{\partial t} = \vec{j} (\vec{r}, t) \]

For simpler notation, textbook authors often drop the argument in the fields and material parameters. It is the context of the problem which determines which of the fields is used.
Complex dielectric constant

\[ \nabla \cdot (\varepsilon_0 \varepsilon \vec{E}) = 0 \quad \text{check this!} \]
\[ \nabla \cdot (\mu_0 \mu \vec{H}) = 0 \]
\[ \nabla \times \vec{E} - i \omega \mu_0 \mu \vec{H} = 0 \]
\[ \nabla \times \vec{H} + i \omega \varepsilon_0 \varepsilon \vec{E} = 0 \]

\[ \varepsilon \leftarrow \left( \varepsilon + \frac{i \sigma}{\omega \varepsilon_0} \right) \]

\[ \nabla \cdot \vec{D}(\vec{r}) = \rho(\vec{r}) \]
\[ \nabla \cdot \vec{B}(\vec{r}) = 0 \]
\[ \nabla \times \vec{E}(\vec{r}) - i \omega \vec{B}(\vec{r}) = 0 \]
\[ \nabla \times \vec{H}(\vec{r}) + i \omega \vec{D}(\vec{r}) = \vec{j}(\vec{r}) \]

\[ \varepsilon_0 \varepsilon \vec{E} \quad \sigma \vec{E} \]

no source current!

\[ \nabla \times \vec{H} + i \omega \varepsilon_0 \left( \varepsilon + \frac{i \sigma}{\omega \varepsilon_0} \right) \vec{E} = 0 \]

we will call it “complex” dielectric constant and denote again with \( \varepsilon \)
Complex dielectric constant

\[ \nabla \cdot \left( \varepsilon_0 \varepsilon \vec{E} \right) = 0 \]
\[ \nabla \cdot \left( \mu_0 \mu \vec{H} \right) = 0 \]
\[ \nabla \times \vec{E} - i \omega \mu_0 \mu \vec{H} = 0 \]
\[ \nabla \times \vec{H} + i \omega \varepsilon_0 \varepsilon \vec{E} = 0 \]

\[ \nabla \cdot \vec{D}(\vec{r}) = \rho(\vec{r}) \]
\[ \nabla \cdot \vec{B}(\vec{r}) = 0 \]
\[ \nabla \times \vec{E}(\vec{r}) - i \omega \vec{B}(\vec{r}) = 0 \]
\[ \nabla \times \vec{H}(\vec{r}) + i \omega \vec{D}(\vec{r}) = \vec{j}(\vec{r}) \]

\[ \varepsilon \leftarrow \left( \varepsilon + \frac{i \sigma}{\omega \varepsilon_0} \right) \]
\[ \varepsilon_0 \varepsilon \vec{E} \]
\[ \sigma \vec{E} \]

no source current!

- Just handy convention – it will be used here.
- The formulation does not distinguish between conduction currents (free charges) and polarization currents (bound charges).
- Energy dissipation is associated with the imaginary part of the dielectric constant.
\[ \vec{\nabla} \cdot (\varepsilon_0 \varepsilon \vec{E}) = 0 \]
\[ \vec{\nabla} \cdot (\mu_0 \mu \vec{H}) = 0 \]
\[ \vec{\nabla} \times \vec{E} - i \omega \mu_0 \mu \vec{H} = 0 \]
\[ \vec{\nabla} \times \vec{H} + i \omega \varepsilon_0 \varepsilon \vec{E} = 0 \]

check that they follow from ...

... independent Eqs.

we will solve them
TE and TM solutions (modes)

assume homogeneity in $y$-direction $\frac{\partial}{\partial y} = 0$

\[
\nabla \times \vec{E} - i\omega \mu_0 \mu \vec{H} = 0
\]
\[
\nabla \times \vec{H} + i\omega \varepsilon_0 \varepsilon \vec{E} = 0
\]
\[
\Rightarrow \text{ two independent sets of equations:}
\]

\[
\frac{\partial E_y}{\partial z} = -i\omega \mu_0 \mu H_x
\]
\[
\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = i\omega \mu_0 \mu H_y
\]
\[
\frac{\partial E_y}{\partial x} = i\omega \mu_0 \mu H_z
\]
\[
\frac{\partial H_y}{\partial z} = i\varepsilon_0 \varepsilon E_x
\]
\[
\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = -i\omega \varepsilon_0 \varepsilon E_y
\]
\[
\frac{\partial H_y}{\partial x} = -i\omega \varepsilon_0 \varepsilon E_z
\]

which have two independent sets of solutions

**TE solution:** \( \vec{E} = (0, E_y, 0) \)
\( \vec{H} = (H_x, 0, H_z) \)

**TM solution:** \( \vec{E} = (E_x, 0, E_z) \)
\( \vec{H} = (0, H_y, 0) \)

Symmetry:
\( \vec{E} \leftrightarrow \vec{H} \)
\( \varepsilon_0 \varepsilon \leftrightarrow -\mu_0 \mu \)
• Maxwell’s equations
• Plane waves
• Pulses and group velocity
• Polarization of light
• Transmission and reflection at an interface
Homogeneous medium: plane wave

\[ \nabla \cdot (\varepsilon_0 \varepsilon \vec{E}) = 0 \]
\[ \nabla \cdot (\mu_0 \mu \vec{H}) = 0 \]
\[ \nabla \times \vec{E} - i \omega \mu_0 \mu \vec{H} = 0 \]
\[ \nabla \times \vec{H} + i \omega \varepsilon_0 \varepsilon \vec{E} = 0 \]

The magnitude of the wavevector (\( \vec{k} \)) is related to the angular frequency by the dispersion equation.

\[ \vec{k} \cdot \vec{E}_0 = 0 \]
\[ \vec{k} \cdot \vec{H}_0 = 0 \]
\[ \vec{k} \times \vec{E}_0 = \omega \mu_0 \mu \vec{H}_0 \]
\[ -\vec{k} \times \vec{H}_0 = \omega \varepsilon_0 \varepsilon \vec{E}_0 \]

\[ c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \]
\[ \vec{k} \cdot \vec{k} = \frac{\omega^2}{c^2 \varepsilon \mu} \]

monochromatic plane wave solutions:

\[ \vec{E}(\vec{r}) = \vec{E}_0 \exp(i \vec{k} \cdot \vec{r}) \]
\[ \vec{H}(\vec{r}) = \vec{H}_0 \exp(i \vec{k} \cdot \vec{r}) \]

Significance of plane waves: Any solution of the Eqs. can be expressed as linear combination of plane waves.
Homogeneous medium: plane wave

when $\vec{E}$, $\vec{H}$, $\vec{k}$ are real vectors:

Define

refractive index $n = \sqrt{\varepsilon \mu} = \sqrt{\varepsilon}$

free-space wavenumber $k_0 = \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda}$

$\vec{k} \cdot \vec{E}_0 = 0$

$\vec{k} \cdot \vec{H}_0 = 0$

$\vec{k} \times \vec{E}_0 = \omega \mu_0 \mu \vec{H}_0$

$-\vec{k} \times \vec{H}_0 = \omega \varepsilon_0 \varepsilon \vec{E}_0$

(alternative forms of dispersion Eq.)

$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$

$\frac{\vec{k} \cdot \vec{k}}{c^2} \varepsilon \mu$

$\vec{k} \cdot \vec{k} = k_0^2 n^2$

Behavior velocity $v = \frac{c}{n}$

(when $n$ is real)
Homogeneous medium: plane wave

\[ \vec{k} = \vec{k}' + ik'' \quad \text{(vectors are generally complex)} \]

\[ \vec{E}(\vec{r'}) = \vec{E}_0 \exp(ik' \cdot \vec{r'}) = \vec{E}_0 \exp(ik' \cdot \vec{r'}) \exp(-k'' \cdot \vec{r'}) \]

Surfaces of constant amplitude are planes perpendicular to \(\vec{k}''\). The amplitude decreases in direction of \(\vec{k}''\).

Wavefronts (=surfaces of equal phase) are perpendicular to \(\vec{k}'\).
Homogeneous medium: plane wave

dispersion Eq. \( \vec{k} \cdot \vec{k} = k_0^2 n^2 \)

\[
\begin{align*}
\vec{k}' \cdot \vec{k}' - \vec{k}'' \cdot \vec{k}'' &= k_0^2 [(n')^2 - (n'')^2] \\
\vec{k}' \cdot \vec{k}'' &= k_0^2 n' n'' \\
\text{looks complicated 😞}
\end{align*}
\]

however, 2 important solutions are simple 😊

\( \vec{k}' \parallel \vec{k}'' \)

The homogeneous wave – occurs for complex \( n \)

\( k' + ik'' = k_0 (n' + in'') \)

choose \( k \) in \( z \) direction =>

\[
\vec{E}(z) = \vec{E}_0 \exp(ik_0 n' z) \exp(-k_0 n'' z)
\]

- define phase velocity
- acts as a “normal” \( n \) loss
  (or gain)

\( n'' = 0 \iff \vec{k}' \perp \vec{k}'' \)

The evanescent wave

- \( k \) is complex even for a purely real refractive index \( n \), i.e., \( n'' = 0 \)
- cannot occur in an infinite homogenous medium (exponential growth)
- we will disuse together with total internal reflection

Evanescent fields play a central role in nano-optics.
Define all the material properties
- Frequency dependent
- Transparent materials can be described by a purely real refractive index $n$
- Absorbing materials (or materials with gain): $n = n' + in''$ is complex

$$\vec{E}(z) = \vec{E}_0 \exp(ik_0 n' z) \exp(-k_0 n'' z)$$
- define phase velocity
- acts as a “normal” $n$ loss (or gain)

valid for the most transparent media

$$n = \sqrt{\varepsilon \mu} = \sqrt{\varepsilon}$$
$$\varepsilon = n^2$$
$$\varepsilon' + i\varepsilon' = (n')^2 - (n'')^2 + i2n'n''$$

Figure 5.5-1 The spectral bands within which selected optical materials transmit light.
Figure 5.5-2  Wavelength dependence of the refractive index of (a) selected crystalline solids (from W. L. Wolfe and G. J. Zissis, Eds., The Infrared Handbook, Environmental Research Institute of Michigan, Ann Arbor, MI, 1978); (b) selected glasses (from W. D. Kingery, H. K. Bowen, and D. R. Uhlmann, Introduction to Ceramics, Wiley, New York, 1976).
The (instantaneous) Poynting vector

\[ \vec{S}(\vec{r}, t) = \vec{E}(\vec{r}, t) \times \vec{H}(\vec{r}, t) \]

- the direction and magnitude of the energy flux
- the energy flux = the power per unit area [W/m\(^2\)] carried by the field

The time-averaged Poynting vector

\[ \langle \vec{S}(\vec{r}) \rangle = \frac{1}{2} \text{Re}\{\vec{E}(\vec{r}) \times \vec{H}^*(\vec{r})\} \]

Photodetection measurements are slow compared with the oscillation period of a wave. Therefore the measured quantity is a time average \( \langle \ldots \rangle \)

\[ e^{-i\omega t} \]

dependence assumed

\[ \langle P \rangle = \int \langle \vec{S} \rangle \cdot \hat{n} \, dA \]

- power [W] = the energy flux through the given surface
- a unit vector normal to the surface

\[ I = \langle \vec{S} \rangle \cdot \hat{n} \]

Intensity (power flux, irradiance)
The time-averaged Poynting vector
\[ \langle \vec{S}(\vec{r}') \rangle = \frac{1}{2} \text{Re} \{ \vec{E}(\vec{r}') \times \vec{H}^*(\vec{r}') \} \]

For a monochromatic plane wave
\[ \vec{E}(\vec{r}') = \vec{E}_0 \exp(i\vec{k} \cdot \vec{r}') \]
\[ \vec{H}(\vec{r}') = \vec{H}_0 \exp(i\vec{k} \cdot \vec{r}') \]
\[ \vec{k} \times \vec{E}_0 = \omega \mu_0 \mu \vec{H}_0 \]

Photodetection measurements are slow compared with the oscillation period of a wave. Therefore the measured quantity is a time average \( \langle \ldots \rangle \)
\[ e^{-i\omega t} \text{ dependence assumed} \]

\[ \langle \vec{S}(\vec{r}') \rangle = \frac{e^{-2\vec{k}'' \cdot \vec{r}}}{2\omega \mu_0} |\vec{E}_0|^2 \text{ Re} \left\{ \frac{\vec{k}}{\mu} \right\} \]

Often \( I \propto |\vec{E}|^2 \)
(describe conditions)
A ray is a line drawn in space corresponding to the direction of flow of radiant energy. In practice we can produce beams (pencils) of light and we can imagine a ray as the limit on the narrowness of such a beam. In homogenous isotropic materials, rays are straight lines parallel to k-vector.

Geometrical optics is an approximate method that describes light propagation in terms of rays. It is applicable when dimensions of obstacles (lenses, mirrors) and also widths of beams $\gg$ wavelength.

**Figure 2.2-4** A spherical wave may be approximated at points near the $z$ axis and sufficiently far from the origin by a paraboloidal wave. For very far points, the spherical wave approaches the plane wave.
• Maxwell’s equations
• Plane waves
• Pulses and group velocity
• Polarization of light
• Transmission and reflection at an interface
Optical pulse (wave packet)

- Monochromatic waves are idealizations never strictly realized in practice.

- "Any" wave can be expressed as a superposition of monochromatic waves

\[
\vec{E}(\vec{r}, t) = \int_{-\infty}^{\infty} \vec{E}(\vec{r}, \omega) e^{-i\omega t} d\omega \quad \text{(IFT)}
\]

found by solving the Maxwell’s Eqs. in the frequency domain

\[
\vec{E}(\vec{r}, \omega) = \vec{E}^*(\vec{r}, -\omega)
\]

For monochromatic plane waves propagating in z

\[
\vec{E}(\vec{r}, \omega) \equiv \hat{x}E(z, \omega) = \hat{x}E(\omega) \exp[i k(\omega) z] \quad n(\omega) = n^*(-\omega)
\]

\[
k(\omega) = \frac{\omega}{c} n(\omega)
\]
Optical pulse (wave packet)

\[
E(z, t) = \int_{-\infty}^{\infty} E(\omega) \exp \left[ -i\omega t + ik(\omega)z \right] d\omega
\]

(pulse = superposition of monochromatic plane waves)

\[
\vec{E}(\vec{r}, t) = \int_{-\infty}^{\infty} \vec{E}(\vec{r}, \omega) e^{-i\omega t} d\omega
\]

\[
\vec{E}(\vec{r}, \omega) \equiv \hat{x} E(z, \omega) = \hat{x} E(\omega) \exp \left[ ik(\omega)z \right]
\]

\[
k(\omega) = \frac{\omega}{c} n(\omega)
\]
The longer the pulse in time, the narrower the spread of the spectra in the frequency domain.

**Optical pulse (wave packet)**

\[
E(z, t) = \int_{-\infty}^{\infty} E(\omega) \exp[-i\omega t + ik(\omega)z] \, d\omega
\]

sharply peaked around \(\omega_0\)

\[
\vec{E}(0, t) = \int_{-\infty}^{\infty} \vec{E}(\omega) \exp(-i\omega t) \, d\omega
\]

\[
\vec{E}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \vec{E}(0, t) \exp(i\omega t) \, dt
\]

**Example:** an idealized pulse with duration \(\Delta t\) and frequency \(\omega_0\)

The longer the pulse in time, the narrower the spread of the spectra in the frequency domain.
Propagation of optical pulse

\[ E(z, t) = \int_{-\infty}^{\infty} E(\omega) \exp[-i\omega t + ik(\omega)z] \, d\omega \]  

(pulse = superposition of monochromatic plane waves)

\[ v(\omega) = \frac{c}{n(\omega)} \]

In a dispersive medium, each monochromatic plane propagates at its own phase velocity 
=> pulse becomes distorted (=dispersion)

\[ v(\omega) = \text{const} \]
no dispersion

\[ v(\omega) \neq \text{const} \]
weak dispersion
Propagation of optical pulse

\[ E(z, t) = \int_{-\infty}^{\infty} E(\omega) \exp[-i\omega t + ik(\omega)z] \, d\omega \]

(pulse = superposition of monochromatic plane waves)

\[ k(\omega) = k(\omega_0) + \left. \frac{dk}{d\omega} \right|_{\omega_0} \Delta \omega + ... \]

\[ \Delta \omega = \omega - \omega_0 \]

(small)

assume it is real for simplicity

\[ E(z, t) = \exp[-i\omega_0 t + ik(\omega_0)z] \int_{(\Delta \omega)} B(\omega_0 + \Delta \omega) \exp \left[ -i\Delta \omega \left( t - \left. \frac{dk}{d\omega} \right|_{\omega_0} z \right) \right] d\Delta \omega + \text{c.c.} \]

(fast varying) phase factor

slowly varying envelope that travels with a velocity

\[ v_g = \frac{d\omega}{dk} \]

(group velocity)
Frequency dispersion in groups of gravity waves on the surface of deep water. The red dot moves with the **phase velocity**, and the green dots propagate with the **group velocity**. In this deep-water case, the phase velocity is twice the group velocity. The red dot overtakes two green dots when moving from the left to the right of the figure.

http://en.wikipedia.org/wiki/Group_velocity

\[ E(z, t) = \exp \left[ -i \omega_0 t + ik(\omega_0)z \right] \int \frac{B(\omega_0 + \Delta \omega)}{(\Delta \omega)} \exp \left[ -i \Delta \omega \left( t - \frac{dk}{d\omega} \right) \right] d\Delta \omega + \text{c.c.} \]

- **(fast varying) phase factor**
- **slowly varying envelope that travels with a velocity**

\[ v_g = \frac{d\omega}{dk} \] (group velocity)
Three velocities

\[ \vec{v} = \frac{\omega}{k} \hat{k} \quad \text{(phase velocity)} \]

\[ \hat{k} := \frac{\vec{k}}{k} \]
- lies in the direction of the wavevector

\[ \vec{v}_g = \vec{\nabla}_k \omega \quad \text{(group velocity)} \]
- perpendicular to phase fronts

\[ \vec{v}_E = \frac{\langle S \rangle}{\langle w \rangle} \quad \text{(energy velocity)} \]

energy density \( w = \frac{1}{2} (\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H}) \) not valid in dispersive media!

more details: [Novotny and Hecht] sect. 2.11;
or [L.D. Landau, E.M. Lifshitz, Electrodynamics of Continuous Media, Pergamon Press (1960)]

The velocities need to be carefully examined to avoid wrong interpretations!
• Maxwell’s equations
• Plane waves
• Pulses and group velocity
• Polarization of light
• Transmission and reflection at an interface
Polarization of light

\[ \vec{E}(\vec{r}, t) = \vec{E}_0 \exp(-i\omega t + i\vec{k} \cdot \vec{r}) \]

Figure 6.1 Various states of polarization (SOP). (a) Linearly (horizontally) polarized. (b) Right-handed circularly polarized. (c) Left-handed circularly polarized. (d) Depolarized.
Polarization of light

when incident light is unpolarized

incident ray, intensity $I_0$

unpolarized light

polarizing sheet

linearly polarized

$\mathbf{E}_y = E \cos \theta$

$I = I_0 \cos^2 \theta$

when incident light (with intensity $I_0$) is linearly polarized
Polarization of light

$I_1 = \frac{1}{2} I_0$

$I_2 = I_1 \cos^2 \theta$

$I_2 = 0$ when $\theta = 90^0$ (crossed polarizers)
The polarizer sheet absorbs radiation polarized in a direction parallel to the long molecules; radiation perpendicular to them passes through.

**Figure 6.22** Dichroic linear polarizer sheet. (a) Extinction. (b) Transmission.
no filter

with polarizer

Polarization of light

\[ \vec{E}(\vec{r}, t) = \vec{E}_0 \exp(-i\omega t + i\vec{k} \cdot \vec{r}) \]
Elliptically polarized wave

\[ \vec{E}(\vec{r}, t) = \vec{E}_0 \exp(-i\omega t + i\vec{k} \cdot \vec{r}) \]

\[ E_x = E_{0x} e^{-i\omega t + ikz} \]

\[ E_y = E_{0y} e^{-i\omega t + ikz} \]

\[ \frac{E_x^2}{a_x^2} + \frac{E_y^2}{a_y^2} - 2 \cos \delta \frac{E_x E_y}{a_x a_y} = \sin^2 \delta \]

\[ \delta = \varphi_y - \varphi_x \]

http://en.wikipedia.org/wiki/Polarization_%28waves%29
Elliptically polarized wave

when $a_x = a_y$

\[ E_x = a_x \cos(-\omega t + kz + \varphi_x) \]
\[ E_y = a_y \cos(-\omega t + kz + \varphi_y) \]

\[ \frac{E_x^2}{a_x^2} + \frac{E_y^2}{a_y^2} - 2 \cos \delta \frac{E_x E_y}{a_x a_y} = \sin^2 \delta \]

\[ \delta = \varphi_y - \varphi_x \]
Outline

- Maxwell’s equations
- Plane waves
- Pulses and group velocity
- Polarization of light
- Transmission and reflection at an interface
Transmission and reflection at an interface

The plane of incidence: defined by the incident ray and a line normal to the interface - here \((x, z)\) plane.

\[
\vec{E}_i(\vec{r}') = \vec{E}_{0i} e^{i\vec{k}_i \cdot \vec{r}} \\
\vec{H}_i(\vec{r}') = \vec{H}_{0i} e^{i\vec{k}_i \cdot \vec{r}} \\
\vec{k}_i = (k_{i_x}, 0, k_{i_z})
\]

(for the chosen coordinate system)

We use continuity of the tangent E- and H-field and obtain:

1) the laws of reflection and refraction
2) relations for the amplitudes of the reflected and transmitted waves (Fresnel reflection and transmission coefficients)
1) Laws of reflection and refraction

The plane of incidence: defined by the incident ray and a line normal to the interface - here \((x,z)\) plane.

\[ \vec{E}_1(\vec{r}') = \vec{E}_{0i} e^{i\vec{k}_i \cdot \vec{r}'} \]
\[ \vec{H}_i(\vec{r}') = \vec{H}_{0i} e^{i\vec{k}_i \cdot \vec{r}'} \]
\[ \vec{k}_i = (k_{ix}, 0, k_{iz}) \]
(for the chosen coordinate system)

The transverse components of the \(k\)-vectors are conserved (here: \(y\) and \(z\) component) \(\Rightarrow\)

All three \(k\)-vectors lie in the plane of incidence.

\[ \theta_i = \theta_r \]

\[ n_1 \sin \theta_i = n_2 \sin \theta_t \]

and ...
1) Laws of reflection and refraction

Transmitted (refracted) light

\[ \vec{k}_t = (k_{tx}, 0, k_{tz}) \]

Reflected light

\[ \vec{k}_r = (-k_{rx}, 0, k_{rz}) \]

\[ \vec{E}_i(\vec{r}) = \vec{E}_{0i} \exp(i\vec{k}_i \cdot \vec{r}) \]

\[ \vec{H}_i(\vec{r}) = \vec{H}_{0i} \exp(i\vec{k}_i \cdot \vec{r}) \]

\[ \vec{k}_i = (k_{ix}, 0, k_{iz}) \]

(for the chosen coordinate system)

... the longitudinal components \((x)\) of the \(k\)-vectors can be calculated as

\[ k_x^2 + k_y^2 + k_z^2 = k_0^2 n^2 \]  \hspace{1cm} \text{(dispersion Eq.)}

reflected wave \hspace{1cm} k_i = k_r \hspace{1cm} k_{rx} = -k_{ix}

transmitted wave \hspace{1cm} k_{tx} = \sqrt{k_0^2 n_2^2 - k_{tz}^2} = \sqrt{k_0^2 n_2^2 - k_{iz}^2}

\( \text{Re}(k_{tx}) \geq 0, \text{Im}(k_{tx}) \geq 0 \)
\[ n_1 \sin \theta_i = n_2 \sin \theta_t \]
\[ n_1 \sin \theta_i = n_2 \sin \theta_t \]
Chromatic dispersion
Rainbow

sunlight → water drops

→ to observer
2) Fresnel reflection and transmission coefficients

**TE polarization, s, perpendicular**

\[ E_y : \quad E_{0iy} + E_{0ry} = E_{0ty} \]
\[ H_z : \quad \frac{k_{ix}E_{0iy}}{\mu_1} - \frac{k_{ix}E_{0ry}}{\mu_1} = \frac{k_{tx}E_{0ty}}{\mu_2} \]

\[ r \equiv \frac{E_{0ry}}{E_{0iy}} = \frac{k_{ix} \mu_2 - k_{tx} \mu_1}{k_{ix} \mu_2 + k_{tx} \mu_1} \]
\[ t \equiv \frac{E_{0ty}}{E_{0iy}} = 1 + r \]

**TM polarization, p, parallel**

\[ H_y : \quad H_{0iy} + H_{0ry} = H_{0ty} \]
\[ E_z : \quad \frac{k_{ix}H_{0iy}}{\varepsilon_1} - \frac{k_{ix}H_{0ry}}{\varepsilon_1} = \frac{k_{tx}H_{0ty}}{\varepsilon_2} \]

\[ r \equiv \frac{H_{0ry}}{H_{0iy}} = \frac{k_{ix} \varepsilon_2 - k_{tx} \varepsilon_1}{k_{ix} \varepsilon_2 + k_{tx} \varepsilon_1} \]
\[ t \equiv \frac{H_{0ty}}{H_{0iy}} = 1 + r \]
Total internal reflection

$$R \equiv |r|^2$$  
(reflectance and transmittance)

$$R + T = 1$$

For now:  \( \mu = 1 \)
Brewster angle

for TM polarization

\[ r \equiv \frac{H_{0ry}}{H_{0iy}} = \frac{k_{ix} \varepsilon_2 - k_{tx} \varepsilon_1}{k_{ix} \varepsilon_2 + k_{tx} \varepsilon_1} = 0 \]

\[ \theta_{i,B} + \theta_{t,B} = 90^0 \]

\[ \theta_{i,B} = \arctan\left(\frac{n_2}{n_1}\right) \]

The reflected light is fully polarized (TE polarization)
The reflected light is fully polarized (TE polarization)
Total internal reflection

\[ \theta_c = \sin^{-1} \left( \frac{n_2}{n_1} \right) \]

\[ n_1 = 1.5 \]
\[ n_2 = 1.0 \]

\[ n_1 > n_2 \]
$$n_1 > n_2$$
Evanescent wave

Check that this is a pure imaginary number in the case of total internal reflection.

\[ \vec{E}_t(\vec{r}, t) = \vec{E}_0 e^{-i\omega t + i \vec{k}_t \cdot \vec{r}} = \vec{E}_0 e^{-i\omega t + i k_{tx} x + i k_{tz} z} \]

\[ k_{tx} = \sqrt{k_0^2 n_2^2 - k_{tz}^2} = \sqrt{k_0^2 n_2^2 - k_{iz}^2} = i \sqrt{k_{iz}^2 - k_0^2 n_2^2} \]

\[ \text{Re}(k_{tx}) \geq 0, \quad \text{Im}(k_{tx}) \geq 0 \]
**Evanescent wave**

1. The field propagates along the surface (z direction).
2. The field does not propagate in x but rather decays exponentially.
3. No time-averaged energy flux in x.

**Mathematical Representation:**

\[ \vec{E}_t(\vec{r}, t) = \vec{E}_0 e^{-i\omega t - k_{tx} x + i k_{tx} z} \]