A The Dirac distribution

A.1 Definition of the Dirac distribution

The Dirac distribution $\delta(x)$ can be introduced by three equivalent ways.

(1) Dirac [1] defined it by relations

$$\int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1,\ \ \delta(x) = 0 \qquad \text{if} \qquad x \neq 0. \tag{1}$$

The distribution is usually depicted by the arrow of unit length (see Fig. 1).

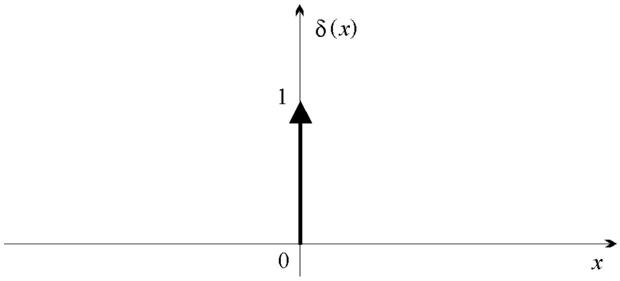

Figure 1: Graph of the Dirac distribution $\delta(x)$.

(2) The "sifting" property of the Dirac distribution may serve as another possible definition: Let as suppose that a funciton $f(x)$ is continnous over the interval $\langle x_1, x_2\rangle$ or that it has at most finite number of finite discontinuities over that interval. Then

$$\int_{x_1}^{x_2} f(x)\,\delta(x-x_0)\,\mathrm{d}x = \begin{cases} \frac{1}{2}\left[f(x_0^-) + f(x_0^+)\right], & \text{if} \quad x_0 \in (x_1, x_2), \\ \frac{1}{2} f(x_0^+), & \text{if} \quad x_0 = x_1, \\ \frac{1}{2} f(x_0^-), & \text{if} \quad x_0 = x_2, \\ 0, & \text{if} \quad x_0 \notin \langle x_1, x_2\rangle. \end{cases} \tag{2}$$

Of course, if the function is continuous, the first of the relations (2) reduces to the form

$$\int_{x_1}^{x_2} f(x)\,\delta(x-x_0)\,\mathrm{d}x = f(x_0), \qquad \text{if} \quad x_0 \in (x_1, x_2), \tag{2a}$$

which is the most frequently appearing form of the sifting property (see Fig. 2).

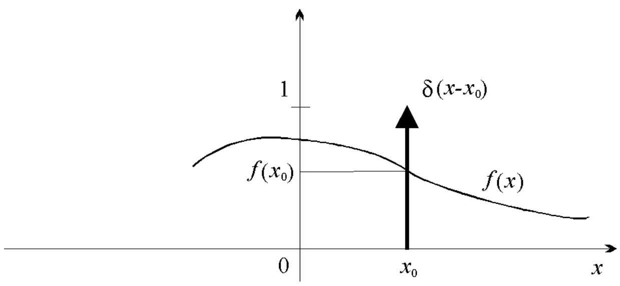

Figure 2: Sifting property of the Dirac distribution.

(3) Very often the Dirac distribution is defined as the limit of the sequence of functions $\delta_p(x)$

$$\delta(x) = \lim_{p\to\infty} \delta_p(x).$$

The function $\delta_p(x)$ have to satisfy two conditions:

$$\lim_{p\to\infty}\int_{-\infty}^{\infty}\delta_p(x)\,\mathrm{d}x = 1 \qquad \text{and} \qquad \lim_{p\to\infty}\frac{\delta_p(x\neq 0)}{\lim_{x\to 0}\delta_p(x)} = 0. \tag{3}$$

In most cases the functions $\delta_p(x)$ satisfy more severe conditions:

$$\int_{-\infty}^{\infty}\delta_p(x)\,\mathrm{d}x = 1 \qquad \text{and} \qquad \lim_{p\to\infty}\delta_p(x\neq 0) = 0. \tag{3a}$$

A.2 Examples of functions $\delta_p(x)$

(a) Probably the most obvious example of functions $\delta_p(x)$ is

$$\delta_p(x) = p\ \mathrm{rect}\,(px) \tag{1}$$

(see Fig. 3). Evidently, these functions satisfy the conditions A.1(3a).

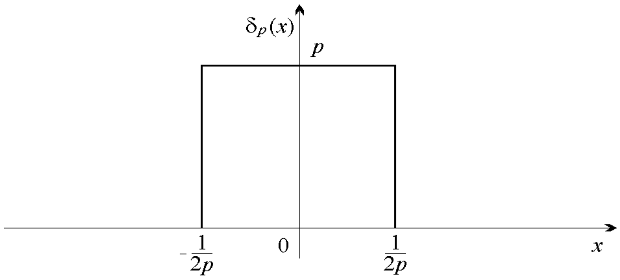

Figure 3: Graph of the function $\delta_p(x) = p\ \mathrm{rect}\,(px)$.

(b) Another obvious example provide the functions

$$\delta_p(x) = p\ \mathrm{tri}\,(px) \tag{2}$$

(see Fig. 4). Also these functions obviously satisfy the conditions A.1(3a).

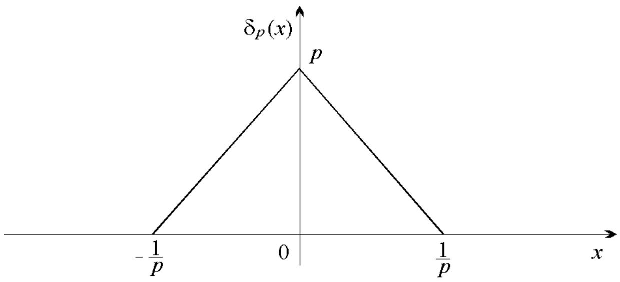

Figure 4: Graph of the function $\delta_p(x) = p\ \mathrm{tri}\,(px)$.

(c) An important example is the sequence of functions

$$\delta_p(x) = \sqrt{\frac{p}{\pi}}\,\exp(-px^2) \tag{3}$$

(see Fig. 5). Let us show, that also these functions satisfy the conditions A.1(3a):

$$\int_{-\infty}^{\infty}\delta_p(x)\,\mathrm{d}x = \sqrt{\frac{p}{\pi}}\int_{-\infty}^{\infty}\exp(-px^2)\,\mathrm{d}x = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp(-t^2)\,\mathrm{d}t = 1.$$

(The integral $I = \int_{-\infty}^{\infty}\exp(-t^2)\,\mathrm{d}t$ is evaluated as follows:

$$I = 2\int_0^{\infty}\exp(-x^2)\,\mathrm{d}x,$$

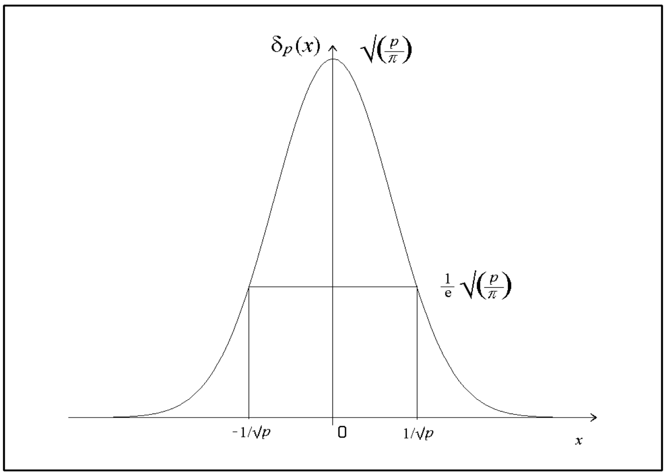

Figure 5: Graph of the function $\delta_p(x) = \sqrt{(p/\pi)}\ \exp(-px^2)$.

so that

$$\begin{aligned} I^2 &= 4\int_0^\infty \exp(-x^2)\,\mathrm{d}x \int_0^\infty \exp(-y^2)\,\mathrm{d}y = 4\int_0^\infty\int_0^\infty \exp\big[-(x^2+y^2)\big]\,\mathrm{d}x\,\mathrm{d}y = \\ &= 4\int_0^\infty\int_0^{\pi/2} \exp(-r^2)r\,\mathrm{d}\varphi\,\mathrm{d}r = 2\pi\int_0^\infty \exp(-r^2)r\,\mathrm{d}r = \\ &= \pi\int_0^\infty \exp(-s)\,\mathrm{d}s = \pi. \end{aligned}$$

Hence $I = \sqrt{\pi}$.) The second condition A.1(3a) is satisfied as well:

$$\lim_{p\to\infty} \delta_p(x \neq 0) = \frac{1}{\sqrt{\pi}} \lim_{p\to\infty} \frac{\sqrt{p}}{\exp(px^2)}\Bigg|_{x\neq 0} = \frac{1}{2x^2\sqrt{\pi}} \lim_{p\to\infty} \frac{\exp(-px^2)}{\sqrt{p}}\Bigg|_{x\neq 0} = 0.$$

(d) Also the functions

$$\delta_p(x) = \frac{1}{\pi}\frac{p}{1+p^2x^2} \tag{4}$$

(see Fig. 6) satisfy the conditions A.1(3a):

$$\int_{-\infty}^{\infty} \delta_p(x)\,\mathrm{d}x = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{p\ \mathrm{d}x}{1+p^2x^2} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{1+t^2} = \frac{1}{\pi}\operatorname{arctg} t\,\Bigg|_{t=-\infty}^{t=\infty} = 1.$$

$$\lim_{p\to\infty} \delta_p(x\neq 0) = \frac{1}{\pi}\lim_{p\to\infty} \frac{p}{1+p^2x^2}\Bigg|_{x\neq 0} = \frac{1}{\pi}\lim_{p\to\infty}\frac{1}{2px^2}\Bigg|_{x\neq 0} = 0.$$

(e) In calculations and proofs of theorems about the Fourier transform we often meet formally different expressions of the function

$$\delta_p(x) = \frac{1}{2\pi}\int_{-p}^{p} \exp(\pm\mathrm{i}tx)\,\mathrm{d}t = \frac{1}{\pi}\int_0^p \cos tx\,\mathrm{d}t = \frac{1}{\pi}\frac{\sin px}{x} = \frac{p}{\pi}\frac{\sin px}{px} \tag{5}$$

(see Fig. 7). The first of the conditions A.1(3a) is satisfied:

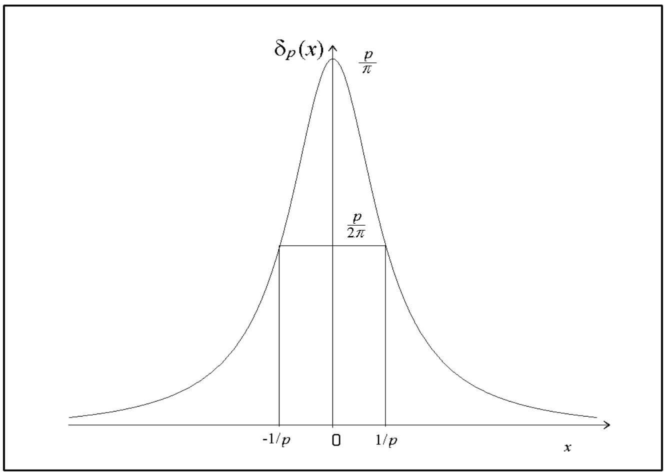

Figure 6: Graph of the function $\delta_p(x) = \frac{1}{\pi}\frac{p}{1+p^2x^2}$.

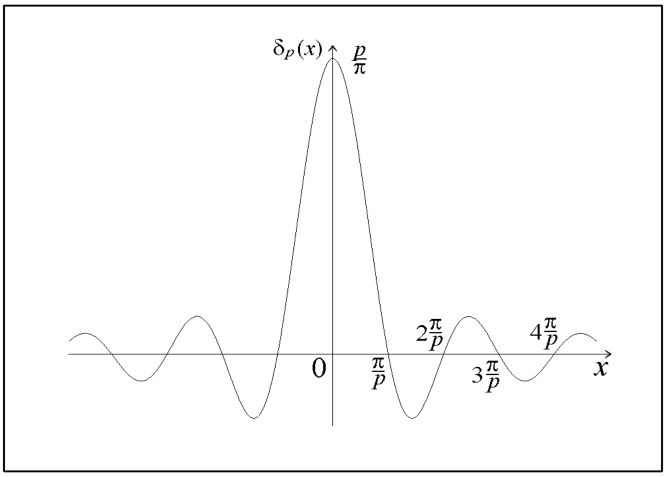

Figure 7: Graph of the function $\delta_p(x) = \frac{\sin px}{\pi x}$.

$$\begin{aligned}\int_{-\infty}^{\infty} \delta_p(x)\,\mathrm{d}x &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin px}{x}\,\mathrm{d}x = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin y}{y}\,\mathrm{d}y = \\ &= \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin y}{y}\,\mathrm{d}y = \frac{2}{\pi}\frac{\pi}{2} = 1\end{aligned}$$

(cf. e.g. [2], 3.721, [3], 5.2.25). The second condition A.1(3a) is however, not satisfied because the corresponding limit does not exist: If $x \neq 0$, the function $\delta_p(x) = \frac{\sin px}{\pi x}$ takes the values from the interval $\langle -\frac{1}{\pi x}, \frac{1}{\pi x}\rangle$, which does not depend on p. The condition A.1(3) is, of course, satisfied because $\lim_{x\to 0}\delta_p(x) = \frac{p}{\pi}$, and hence

$$\lim_{p\to\infty} \frac{\delta_p(x\neq 0)}{\lim_{x\to 0}\delta_p(x)} = \lim_{p\to\infty} \frac{\sin px}{\pi x}\frac{\pi}{p}\Bigg|_{x\neq 0} = \lim_{p\to\infty}\frac{\sin px}{px}\Bigg|_{x\neq 0} = 0.$$

A.3 Properties of the Dirac distribution

(a) Let us denote by x_n the roots of the equation $f(x) = 0$ and suppose that $f'(x_n) \neq 0$. Then

$$\delta(f(x)) = \sum_n \frac{\delta(x - x_n)}{|f'(x_n)|}. \tag{1}$$

Proof: Let us choose the numbers a_n, b_n, in a neighbourhood of each root x_n in such a way that $a_n < x_n < b_n$ and the function $f(x)$ is monotonous in the interval $< a_n, b_n >$. Then

$$\int\limits_{-\infty}^{\infty} g(x)\delta(f(x))\,\mathrm{d}x = \sum_n I_n, \tag{2}$$

where

$$I_n = \int\limits_{a_n}^{b_n} g(x)\delta(f(x))\,\mathrm{d}x = \int\limits_{a_n}^{b_n} g(x)\delta((x - x_n)f'(x_n))\,\mathrm{d}x. \tag{3}$$

By the substitution $(x - x_n)f'(x_n) = t$ we get

$$I_n = \frac{1}{f'(x_n)} \int\limits_{(a_n - x_n)f'(x_n)}^{(b_n - x_n)f'(x_n)} g\left(\frac{t}{f'(x_n)} + x_n\right)\delta(t)\,\mathrm{d}t. \tag{4}$$

If $f'(x_n) < 0$, the upper limit of integration is greater than the lower one and it is

$$I_n = -\frac{1}{f'(x_n)} \int\limits_{(b_n - x_n)f'(x_n)}^{(a_n - x_n)f'(x_n)} g\left(\frac{t}{f'(x_n)} + x_n\right)\delta(t)\,\mathrm{d}t = \frac{g(x_n)}{|f'(x_n)|}.$$

From (2) and (3) and from the sifting property of the Dirac distribution A.1(2a) it follows

$$\int\limits_{-\infty}^{\infty} g(x)\delta(f(x))\,\mathrm{d}x = \sum_n \frac{g(x_n)}{|f'(x_n)|} = \sum_n \frac{1}{|f'(x_n)|} \int\limits_{a_n}^{b_n} g(x)\delta(x - x_n)\,\mathrm{d}x. \tag{5}$$

If $f'(x_n) > 0$, the relation (5) follows imediately from (2), (4) and A.1(2a). Thus, the equation (1) is proved.

Important consequences of equation (1) are:

$$\delta(-x) = \delta(x), \tag{6}$$

$$\delta(ax - x_0) = \frac{1}{|a|}\,\delta\left(x - \frac{x_0}{a}\right), \tag{7}$$

$$\delta\left(\sin \pi\frac{x}{a}\right) = \frac{|a|}{\pi} \sum_{m=-\infty}^{\infty} \delta(x - ma), \tag{8}$$

$$\delta(x^2 - a^2) = \frac{\delta(x - a) + \delta(x + a)}{2|a|}. \tag{9}$$

(b) It is

$$\delta(x) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{|x|}\right), \tag{10}$$

i.e.

$$\delta(x) = \frac{\mathrm{d}H(x)}{\mathrm{d}x}, \tag{11}$$

where

$$H(x) = \frac{1}{2}\left(1 + \frac{x}{|x|}\right) \tag{12}$$

is the Heaviside function.

Proof:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{|x|}\right) = \frac{1}{2}\lim_{p\to\infty}\frac{\mathrm{d}}{\mathrm{d}x}\frac{2}{\pi}\,\mathrm{arctg}\,px = \frac{1}{\pi}\lim_{p\to\infty}\frac{p}{1+p^2x^2} = \delta(x) \tag{13}$$

(cf. A.2(4)).

(c) The following properties of the Dirac distribution are frequently used (e.g. while evaluating convolutions and cross–corelations):

$$f(x)\,\delta(x-a) = f(a)\,\delta(x-a), \tag{14}$$

$$\int_c^d \delta(x-a)\,\delta(x-b)\;\mathrm{d}x = \delta(a-b), \qquad c < \min(a,\,b), \quad d > \max(a,\,b). \tag{15}$$

A.4 The Dirac distribution obtained from complete system of orthonormal functions

Interesting and often useful expressions of the Dirac distribution can be obtained from complete systems of orthogonal functions.

Let functions $\psi_n(x)$, n being integers, form a complete orthonormal system of functions on an interval $\langle x_1, x_1+a\rangle$ and let x and x_0 be inner points of that interval. Then

$$\sum_n \psi_n^*(x)\,\psi_n(x_0) = \delta(x-x_0), \tag{1}$$

where the summation goes over all n for which the orthonormal system $\{\psi_n(x)\}$ is complete.

Proof: To prove (1) we shall demonstrate that the left–hand side of equation (1) has the sifting property of the Dirac distribution

$$\int_{x_1}^{x_1+a} f(x)\,\delta(x-x_0)\;\mathrm{d}x = f(x_0),$$

i.e. that

$$\int_{x_1}^{x_1+a} f(x)\sum_n \psi_n^*(x)\,\psi_n(x_0)\;\mathrm{d}x = f(x_0). \tag{2}$$

To prove (2) we expand function $f(x)$ into the system of orthonormal functions $\{\psi_n(x)\}$, i.e.

$$f(x) = \sum_m c_m\,\psi_m(x), \tag{3}$$

where

$$c_m = \int_{x_1}^{x_1+a} f(x)\,\psi_m^*(x)\;\mathrm{d}x.$$

Now we insert the series (3) into the left–hand side of equation (2), exchange the order of integration and addition and make use of the condition of orthonormality

$$\int_{x_1}^{x_1+a} \psi_n^*(x)\,\psi_m(x)\;\mathrm{d}x = \delta_{m,n}:$$

$$\begin{aligned}
\int_{x_1}^{x_1+a} \sum_m c_m\,\psi_m(x)\sum_n \psi_n^*(x)\,\psi_n(x_0)\;\mathrm{d}x &= \sum_m c_m \sum_n \psi_n(x_0)\int_{x_1}^{x_1+a} \psi_n^*(x)\,\psi_m(x)\,\mathrm{d}x = \\
&= \sum_m c_m \sum_n \psi_n(x_0)\,\delta_{m,n} = \\
&= \sum_m c_m\,\psi_m(x_0) = f(x_0).
\end{aligned}$$

Thus, we have got the right–hand side of (2) and the statement (1) is proved.

The functions

$$\psi_n(x) = \frac{1}{\sqrt{|a|}} \exp\left(\mathrm{i}n2\pi\,\frac{x}{a}\right), \qquad n = 0, \pm1, \pm2, \ldots$$

form the complete orthonormal system on any interval of the length $|a|$ and hence also on the interval $\langle -a/2, a/2\rangle$. Therefore, according to (1) it is

$$\frac{1}{|a|}\sum_{n=-\infty}^{\infty} \exp\left(\mathrm{i}n2\pi\,\frac{x-x_0}{a}\right) = \delta(x-x_0), \qquad x, x_0 \in \left(-\frac{a}{2}, \frac{a}{2}\right).$$

Every summand of the infinite geometric series on the left–hand side of the foregoing relation is a periodic function with the period a. Consequently the sum of the series has the same period and for all x, x_0 it holds

$$\frac{1}{|a|}\sum_{n=-\infty}^{\infty} \exp\left(\mathrm{i}n2\pi\,\frac{x-x_0}{a}\right) = \sum_{m=-\infty}^{\infty} \delta(x-x_0-ma). \tag{4}$$

Relation (4) is important for the proof of the fact, that the Fourier transform of the lattice function is proportional to the lattice function characterizing the reciprocal lattice (cf. section 4.3). This is true for the lattices of any dimensions N, N being integer $N \geq 1$. (To be prepared for the proof in the space of the dimension $N \geq 2$ we denote the length of the interval $|a|$, so that a in equation (4) may be both positive and negative.)

The series at the left–hand side of (4) may be rewritten in various forms. For example

$$1 + 2\sum_{n=1}^{\infty} \cos\left(n2\pi\,\frac{x-x_0}{a}\right) = |a| \sum_{m=-\infty}^{\infty} \delta(x-x_0-ma). \tag{5}$$

The series at the left–hand side of (4) is a geometric series of the ratio $\exp\left(\mathrm{i}2\pi\,\frac{x-x_0}{a}\right)$. We may replace it by the limit

$$\sum_{n=-\infty}^{\infty} \exp\left(\mathrm{i}n2\pi\,\frac{x-x_0}{a}\right) = \lim_{p\to\infty} \sum_{n=-p}^{p} \exp\left(\mathrm{i}n2\pi\,\frac{x-x_0}{a}\right).$$

By summing $2p+1$ terms of the limit we get

$$\begin{aligned}
\lim_{p\to\infty} \sum_{n=-p}^{p} \exp\left(\mathrm{i}n2\pi\,\frac{x-x_0}{a}\right) &= \lim_{p\to\infty} \left\{ \exp\left(-\mathrm{i}p2\pi\,\frac{x-x_0}{a}\right) \frac{1-\exp\left[\mathrm{i}(2p+1)2\pi\,\frac{x-x_0}{a}\right]}{1-\exp\left(\mathrm{i}2\pi\,\frac{x-x_0}{a}\right)} \right\} = \\
&= \lim_{p\to\infty} \left\{ \exp\left(-\mathrm{i}p2\pi\,\frac{x-x_0}{a}\right) \frac{\exp\left[\mathrm{i}(2p+1)\pi\,\frac{x-x_0}{a}\right]}{\exp\left(\mathrm{i}\pi\,\frac{x-x_0}{a}\right)} \times \right. \\
&\qquad \left. \times \frac{\exp\left[-\mathrm{i}(2p+1)\pi\,\frac{x-x_0}{a}\right] - \exp\left[\mathrm{i}(2p+1)\pi\,\frac{x-x_0}{a}\right]}{\exp\left(-\mathrm{i}\pi\,\frac{x-x_0}{a}\right) - \exp\left(\mathrm{i}\pi\,\frac{x-x_0}{a}\right)} \right\} = \\
&= \lim_{p\to\infty} \frac{\sin\left[(2p+1)\pi\,\frac{x-x_0}{a}\right]}{\sin\left(\pi\,\frac{x-x_0}{a}\right)}.
\end{aligned}$$

Hence

$$\lim_{p\to\infty} \frac{\sin\left[(2p+1)\pi\,\frac{x-x_0}{a}\right]}{\sin\left(\pi\,\frac{x-x_0}{a}\right)} = |a| \sum_{m=-\infty}^{\infty} \delta\big(x-x_0-ma\big). \tag{6}$$

A.5 The Dirac distribution in E_N

(a) In Cartesian coordinates

$$\int_D \cdots \int f(\vec{x})\delta(\vec{x}-\vec{x}_0)\,\mathrm{d}^N\vec{x} = f(\vec{x}_0), \text{ if } \vec{x}_0 \in D. \tag{1}$$

From the fact that

$$\int_D \cdots \int f(x_1, x_2, \dots, x_N)\delta(x_1 - x_{01})\delta(x_2 - x_{02})\dots\delta(x_N - x_{0N})\,\mathrm{d}x_1\,\mathrm{d}x_2\dots\,\mathrm{d}x_N =$$

$$= f(x_{01}, x_{02}, \dots, x_{0N}),$$

it follows that in Cartesian coordinates

$$\delta(\vec{x}-\vec{x}_0) = \delta(x_1 - x_{01})\delta(x_2 - x_{02})\dots\delta(x_N - x_{0N}) = \prod_{k=1}^{N}\delta(x_k - x_{0k}). \tag{2}$$

Obviously

$$\delta(a\vec{x}) = \frac{1}{|a|^N}\,\delta(\vec{x}). \tag{3}$$

(b) General coordinates

Let the Cartesian coordinates $x_1, x_2, \dots, x_N$ be connected with general coordinates $y_1, y_2, \dots, y_N$ in E_N by relations

$$\begin{aligned} x_1 &= x_1(y_1, \dots, y_N), \\ x_2 &= x_2(y_1, \dots, y_N), \\ &\vdots \\ x_N &= x_N(y_1, \dots, y_N) \end{aligned}$$

with the Jacobian

$$J(y_1, \dots, y_N) = \begin{vmatrix} \frac{\partial x_1}{\partial y_1}, & \dots, & \frac{\partial x_1}{\partial y_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_N}{\partial y_1}, & \dots, & \frac{\partial x_N}{\partial y_N} \end{vmatrix}.$$

If $x_1^{(P)}, x_2^{(P)}, \dots, x_N^{(P)}$ and $y_1^{(P)}, y_2^{(P)}, \dots, y_N^{(P)}$ are coordinates of a point P and if $J(y_1^{(P)}, \dots, y_N^{(P)}) \neq 0$, then

$$\begin{aligned} \delta(x_1 - x_1^{(P)})\delta(x_2 - x_2^{(P)})\dots\delta(x_N - x_N^{(P)}) &= \\ = \frac{1}{|J(y_1, \dots, y_N)|}\,\delta(y_1 - y_1^{(P)})\delta(y_2 - y_2^{(P)})\dots\delta(y_N - y_N^{(P)}). \end{aligned} \tag{4}$$

If, however, $J(y_1^{(P)}, \dots, y_N^{(P)}) = 0$ and the point P is specified by k coordinates $y_1^{(P)}, y_2^{(P)}, \dots, y_k^{(P)}$ (that means that $N-k$ coordinates $y_{k+1}, y_{k+2}, \dots, y_N$ are superfluous for the specification of the point P), we denote by

$$J_k(y_1, \dots, y_k) = \int \cdots \int J(y_1, \dots, y_N)\,\mathrm{d}y_{k+1}\dots\,\mathrm{d}y_N$$

the integral over the $N-k$ superfluous coordinates and it holds

$$\delta(x_1 - x_1^{(P)})\delta(x_2 - x_2^{(P)}) \dots \delta(x_N - x_N^{(P)}) =$$
$$= \frac{1}{|J_k(y_1, \dots, y_k)|} \delta(y_1 - y_1^{(P)})\delta(y_2 - y_2^{(P)}) \dots \delta(y_k - y_k^{(P)}). \tag{5}$$

(c) Example: Polar coordinates in E_2

$$\begin{array}{l} x_1 = r\cos\varphi, \\ x_2 = r\sin\varphi, \end{array} \qquad J(r,\varphi) = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r.$$

(i) At points $P\big(r^{(P)}, \varphi^{(P)}\big)$, $r^{(P)} \neq 0$, it is

$$\delta(x_1 - x_1^{(P)})\delta(x_2 - x_2^{(P)}) = \frac{\delta(r - r^{(P)})\delta(\varphi - \varphi^{(P)})}{r}.$$

(ii) At the point $P \equiv r^{(P)} = 0$ the coordinate φ is superfluous and

$$J_1(r) = \int_\alpha^{\alpha+2\pi} r \,\mathrm{d}\varphi = 2\pi r,$$

so that

$$\delta(x_1)\delta(x_2) = \frac{\delta(r)}{2\pi r}.$$

(d) Example: Spherical coordinates in E_3 (see Fig. 8)

$$\begin{aligned} x_1 &= r\sin\vartheta\cos\varphi, \\ x_2 &= r\sin\vartheta\sin\varphi, \\ x_3 &= r\cos\vartheta, \end{aligned}$$

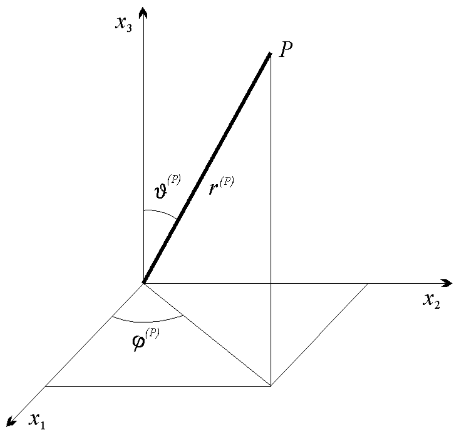

Figure 8: Spherical coordinates.

$$J(r,\vartheta,\varphi) = \begin{vmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \vartheta} & \frac{\partial x_1}{\partial \varphi} \\ \frac{\partial x_2}{\partial r} & \frac{\partial x_2}{\partial \vartheta} & \frac{\partial x_2}{\partial \varphi} \\ \frac{\partial x_3}{\partial r} & \frac{\partial x_3}{\partial \vartheta} & \frac{\partial x_3}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \sin\vartheta\cos\varphi & r\cos\vartheta\cos\varphi & -r\sin\vartheta\sin\varphi \\ \sin\vartheta\sin\varphi & r\cos\vartheta\sin\varphi & r\sin\vartheta\cos\varphi \\ \cos\vartheta & -r\sin\vartheta & 0 \end{vmatrix} =$$
$$= r^2\sin\vartheta.$$

(i) At the point P with coordinates $r^{(P)} \neq 0$, $\vartheta^{(P)} \neq 0$, $\vartheta^{(P)} \neq \pi$, i.e. with $J(r^{(P)}, \vartheta^{(P)}, \ \varphi^{(P)}) \neq 0$, it is

$$\delta(x_1 - x_1^{(P)})\delta(x_2 - x_2^{(P)})\delta(x_3 - x_3^{(P)}) = \frac{\delta(r - r^{(P)})\delta(\vartheta - \vartheta^{(P)})\delta(\varphi - \varphi^{(P)})}{r^2 \sin\vartheta}.$$

(ii) At the point P with coordinates $r^{(P)} \neq 0$, $\vartheta^{(P)} = 0$, or $\vartheta^{(P)} = \pi$, the Jacobian $J(r^{(P)}, \vartheta^{(P)}, \varphi^{(P)}) = 0$ and

$$J_2(r, \vartheta) = \int_0^{2\pi} r^2 \sin\vartheta \, \mathrm{d}\varphi = 2\pi r^2 \sin\vartheta.$$

Therefore

$$\delta(x_1)\delta(x_2)\delta(x_3 - x_3^{(P)}) = \frac{\delta(r - r^{(P)})\delta(\vartheta)}{2\pi r^2 \sin\vartheta}.$$

(iii) At the point P with $r^{(P)} = 0$, it is $J(r^{(P)}, \vartheta^{(P)}, \varphi^{(P)}) = 0$ and

$$J_1(r) = \int_0^{\pi}\int_0^{2\pi} r^2 \sin\vartheta \, \mathrm{d}\varphi \, \mathrm{d}\vartheta = 4\pi r^2.$$

Therefore

$$\delta(x_1)\delta(x_2)\delta(x_3) = \frac{\delta(r)}{4\pi r^2}.$$

(e) Example: Obligue coordinates (important for the Fourier transform of lattices in E_N, $N \geq 2$). Let

$$x_i = a_{ik}\, y_k, \;\; \det \|a_{ik}\| \equiv \det \mathbf{A} \neq 0.$$

More explicitly this can be rewritten as

$$\begin{aligned} x_1 &= a_{11}y_1 + \ldots + a_{1N}y_N, \\ &\vdots \\ x_N &= a_{N1}y_1 + \ldots + a_{NN}y_N, \end{aligned}$$

or in the matrix form

$$\left\| \begin{array}{c} x_1 \\ \vdots \\ x_N \end{array} \right\| = \left\| \begin{array}{ccc} a_{11}, & \ldots, & a_{1N} \\ \vdots & & \vdots \\ a_{N1}, & \ldots, & a_{NN} \end{array} \right\| \left\| \begin{array}{c} y_1 \\ \vdots \\ y_N \end{array} \right\|,$$

i.e.

$$\vec{x} = \mathbf{A}\vec{y}.$$

As

$$\frac{\partial x_i}{\partial y_k} = a_{ik},$$

the Jacobian is

$$J(y_1, \ldots, y_N) = \left| \begin{array}{ccc} a_{11}, & \ldots, & a_{1N} \\ \vdots & & \vdots \\ a_{N1}, & \ldots, & a_{NN} \end{array} \right| = \det \mathbf{A}$$

and

$$\delta(x_1 - x_1^{(P)}) \ldots \delta(x_N - x_N^{(P)}) = \frac{1}{|\det \mathbf{A}|} \delta(y_1 - y_1^{(P)}) \ldots \delta(y_N - y_N^{(P)}),$$

i.e.

$$\delta(\vec{x} - \vec{x}^{(P)}) = \frac{1}{|\det \mathbf{A}|}\, \delta(\vec{y} - \vec{y}^{(P)}).$$

A.6 Notes and features

$$\delta_p(x) = \frac{1}{2} \frac{p}{(1 + p^2 x^2)^{3/2}} \tag{1}$$

$$\delta_p(x) = \frac{2^{2n-3}(n-1)!(n-2)!}{(2n-3)!} \frac{p}{\pi} \frac{1}{(1 + p^2 x^2)^n}, \qquad n = 2, 3, \ldots \tag{2}$$

$$\delta_p(x) = \frac{p}{\pi} \left(\frac{\sin px}{px} \right)^2 \tag{3}$$

$$\delta(y - y_0) = y \int\limits_0^\infty J_m(xy) J_m(xy_0) x \, \mathrm{d}x \tag{4}$$

$$\delta_p(x, y) = \frac{p}{\pi} \exp\{-p[1 - \exp(-x^2 - y^2)]\} \tag{5}$$

$$\delta_p(x, y) = \frac{p^2}{\pi} \operatorname{circ} \left(p\sqrt{x^2 + y^2} \right) \tag{6}$$

$$\delta_p(x, y) = \frac{p}{4\pi} \frac{2\, J_1\left(p\sqrt{x^2 + y^2} \right)}{p\sqrt{x^2 + y^2}} \tag{7}$$

$$\delta(xy) = \frac{\delta(x) + \delta(y)}{\sqrt{x^2 + y^2}} \tag{8}$$

References

[1] Dirac P. A. M.: *The Principles of Quantum Mechanics.* 4th edition. At the Clarendon Press, Oxford 1958, §15.

[2] Gradshteyn I. S., Ryzhik I. M.: *Table of Integrals, Series, and Products.* Academic Press, New York and London 1994.

[3] Abramowitz M., Stegun I. A.: *Handbook of Mathematical Functions.* Dover Publications, Inc., New York 1972.