1 The Fourier transform

1.1 Definition of the Fourier transform

The Fourier transform is defined in different ways in various fields of its application. Consequently, the formulae expressing important theorems (e.g. the convolution theorem, the Rayleigh–Parseval theorem) differ by constants. Neverthleless, the chosen definitions are used more or less consistently in frames of particular disciplines. To get an interdisciplinary overview we involve three constants A, B and k in the definition of the Fourier transform and a special choice of them provides the definitions used in particular disciplines.

Hence, we define the Fourier transform $\mathrm{FT}\{f(\vec{x})\}$ of the function $f(\vec{x})$ and the inverse Fourier transform $\mathrm{FT}^{-1}\{F(\vec{X})\}$ of the function $F(\vec{X})$ by integrals

$$\mathrm{FT}\{f(\vec{x})\} = A^N \int_{-\infty}^{\infty} \cdots \int f(\vec{x}) \exp(-\mathrm{i}k\vec{X}\cdot\vec{x}) \, \mathrm{d}^N\vec{x}, \qquad (1)$$

$$\mathrm{FT}^{-1}\{F(\vec{X})\} = B^N \int_{-\infty}^{\infty} \cdots \int F(\vec{X}) \exp(\mathrm{i}k\vec{X}\cdot\vec{x}) \, \mathrm{d}^N\vec{X}. \qquad (2)$$

Here, it is supposed that $f(\vec{x})$ and $F(\vec{X})$ are absolutely integrable piecewise smooth complex functions of real variables $\vec{x}$, $\vec{X} \in E_N$. The constants A and B may be complex, but k must be real. Nevertheless, it would be only formal and useless to take A and B complex because it would just complicate formulations and proofs of some theorems. Without any lost of generality we may consider the constants A and B to be real and positive. The choice of constants A, B and k is, however, bound by the condition

$$AB = \frac{|k|}{2\pi}, \qquad (3)$$

which follows from the fundamental theorem (cf. section 1.2). The condition (3) guarantees that the functions $f(\vec{x})$ and $F(\vec{X})$ form a Fourier pair, because it follows from (1), (2) and (3) that

$$F(\vec{X}) = \mathrm{FT}\{f(\vec{x})\}, \; f(\vec{x}) = \mathrm{FT}^{-1}\{F(\vec{X})\}. \qquad (4)$$

at points where $f(\vec{x})$ and $F(\vec{X})$ are continuous. Both the integrals in (1) and (2) are called the Fourier integral and the phasors in the integrands are named the Fourier kernel.

As already mentioned, in various disciplines the definitions of the Fourier transform are at variance with the choice of the constants A, B, k. In mathematics the symmetrical form with $A = B = 1$, $k = 2\pi$ is generally used [1], [2], but not at all always (e.g. $A = B = \frac{1}{\sqrt{2\pi}}$, $k = \pm 1$ are used in [3], [4], [5]). In crystallography the form with $A = B = 1$, $k = -2\pi$ prevails [6], [7], but we often find $A = B = 1$, $k = 2\pi$ [8] or even $A = 1$, $B = \frac{1}{2\pi}$, $k = -1$ [9]. In solid state physics, surface physics and circuit theory $A = 1$, $B = \frac{1}{2\pi}$, $k = 1$ or $k = -1$. Attention must be paid to the choice of these constants when using collections of formulae of the Fourier transform. Different choice of the constants also caused that the definitions of the reciprocal lattice used in solid state physics and in crystallography differ by the factor 2π and brought about emotive disputations over it [10].

1.2 The fundamental theorem

Let $f(\vec{x})$ be a function for which the Fourier integral exists. Then at points, where $f(\vec{x})$ is continuous

$$\mathrm{FT}^{-1}\big\{\mathrm{FT}\{f(\vec{x})\}\big\} = \mathrm{FT}\big\{\mathrm{FT}^{-1}\{f(\vec{x})\}\big\} = f(\vec{x}). \qquad (1)$$

At points of ordinary discontinuity the application of the Fourier transform and the inverse Fourier transform provides the mean value of $f(\vec{x})$ in infinitesimal neighbourhood of the point of discontinuity.

The proof is based on the change in the order of integration after which the inner integral gives the Dirac distribution (this is a frequently used trick in the Fourier formalism):

$$\begin{aligned}\mathrm{FT}^{-1}\big\{\mathrm{FT}\{f(\vec{x})\}\big\} &= B^N \int_{-\infty}^{\infty}\!\!\cdots\!\int \left\{ A^N \int_{-\infty}^{\infty}\!\!\cdots\!\int f(\vec{x}')\exp(-\mathrm{i}k\vec{X}\cdot\vec{x}')\,\mathrm{d}^N\vec{x}' \right\} \exp(\mathrm{i}k\vec{X}\cdot\vec{x})\,\mathrm{d}^N\vec{X} = \\ &= (AB)^N \int_{-\infty}^{\infty}\!\!\cdots\!\int f(\vec{x}') \left\{ \int_{-\infty}^{\infty}\!\!\cdots\!\int \exp\left[\mathrm{i}k(\vec{x}-\vec{x}')\cdot\vec{X}\right]\,\mathrm{d}^N\vec{X} \right\} \mathrm{d}^N\vec{x}'. \qquad (2)\end{aligned}$$

The integral in the composed brackets in (2) is the Dirac distribution (cf. A.2(5)):

$$\int_{-\infty}^{\infty}\!\!\cdots\!\int \exp\left[\mathrm{i}k(\vec{x}-\vec{x}')\cdot\vec{X}\right]\,\mathrm{d}^N\vec{X} = \left(\frac{2\pi}{|k|}\right)^N \delta(\vec{x}-\vec{x}'). \qquad (3)$$

Substituting it into (2) we get

$$\mathrm{FT}^{-1}\big\{\mathrm{FT}\{f(\vec{x})\}\big\} = \left(AB\,\frac{2\pi}{|k|}\right)^N \int_{-\infty}^{\infty}\!\!\cdots\!\int f(\vec{x}')\,\delta(\vec{x}-\vec{x}')\,\mathrm{d}^N\vec{x}'. \qquad (4)$$

According to the sifting property of the Dirac distribution the integral in (4) gives $f(\vec{x})$ at the points of continuity of the function $f(\vec{x})$ and the mean value of $f(\vec{x})$ in an infinitesimal neighbourhood of points of regular discontinuity of $f(\vec{x})$. Thus, if we want to get the statement of the fundamental theorem we must choose the constants A, B, k in such a way that

$$\frac{AB}{|k|} = \frac{1}{2\pi},$$

as we have done in the definition of the Fourier transform.

1.3 Examples

If we want to know the Fourier transform of a function we ususally consult dictionaries and tables which supplement some monographs dealing with the Fourier transform (see e.g. [13], [14], [15]) or which are published as a separate book (e.g. [16]). Nevertheless, to get an idea how the tabulated data are obtained and where various restrictions take usually their place of origin we will calculate several examples in detail.

1.3.1 Example: $f(\vec{x}) = \delta(\vec{x}-\vec{a})$.

The Fourier transform of the Dirac distribution follows from the sifting property A.1(2a):

$$\mathrm{FT}\left\{\delta(x-a)\right\} = A\int_{-\infty}^{\infty} \delta(x-a)\exp(-\mathrm{i}kxX)\,\mathrm{d}x = A\exp(-\mathrm{i}kaX). \qquad (1)$$

In the case of the N-dimensional Dirac distribution it is

$$\mathrm{FT}\left\{\delta(\vec{x}-\vec{a})\right\} = A^N \exp(-\mathrm{i}k\vec{a}\cdot\vec{X}). \qquad (2)$$

1.3.2 Example: $f(x) = \exp(\mathrm{i}ax)$.

The Fourier transform of the phasor is proportional to the Dirac distribution as it follows from A.2(5). With the use of A.3(7) and A.3(6) we get

$$\begin{aligned}\mathrm{FT}\left\{\exp(\mathrm{i}ax)\right\} &= A\int_{-\infty}^{\infty} \exp\left[\mathrm{i}(a-kX)x\right]\,\mathrm{d}x = A\lim_{M\to\infty}\int_{-M}^{M} \exp\left[\mathrm{i}(a-kX)x\right]\,\mathrm{d}x = \\ &= 2\pi A\,\delta(a-kX) = \frac{2\pi}{|k|}A\,\delta\left(X-\frac{a}{k}\right) = \frac{1}{B}\,\delta\left(X-\frac{a}{k}\right). \qquad (3)\end{aligned}$$

Particularly, if $a=0$ we get

$$\mathrm{FT}\{1\} = \frac{1}{B}\,\delta(X). \tag{4}$$

In the case of N Cartesian variables it is

$$\mathrm{FT}\{\exp(\mathrm{i}\vec{a}\cdot\vec{x})\} = \frac{1}{B^N}\,\delta\left(\vec{X} - \frac{\vec{a}}{k}\right). \tag{5}$$

If $\vec{a} = \vec{0}$ the relation (5) is reduced to

$$\mathrm{FT}\{1\} = \frac{1}{B^N}\,\delta(\vec{X}). \tag{6}$$

1.3.3 Example: The Gaussian function $f(x) = \exp(-a^2x^2)$

The Fourier transform of the Gaussian function is important in optics, e.g. at the study of the so–called Gaussian beams. Therefore, we shall give here the details of its calculation.

The Gaussian function is even and we can restrict the integration to the positive region of the variable of integration:

$$\mathrm{FT}\left\{\exp(-a^2x^2)\right\} = A\int_{-\infty}^{\infty}\exp(-a^2x^2)\exp(-\mathrm{i}kxX)\,\mathrm{d}x = 2A\int_0^{\infty}\exp\left(-a^2x^2\right)\cos(kxX)\,\mathrm{d}x.$$

Now we expand the cosine into the power series and interchange the order of integration and summation:

$$\begin{aligned}\mathrm{FT}\left\{\exp(-a^2x^2)\right\} &= 2A\int_0^{\infty}\exp(-a^2x^2)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(kxX)^{2n}\,\mathrm{d}x = \\ &= 2A\sum_{n=0}^{\infty}\frac{(-1)^n(kX)^{2n}}{(2n)!}\int_0^{\infty}x^{2n}\exp(-a^2x^2)\,\mathrm{d}x.\end{aligned} \tag{7}$$

By integration by parts we get

$$I_n = \int_0^{\infty}x^{2n}\exp(-a^2x^2)\,\mathrm{d}x = \frac{2a^2}{2n+1}\int_0^{\infty}x^{2(n+1)}\exp(-a^2x^2)\,\mathrm{d}x = \frac{2a^2}{2n+1}I_{n+1}.$$

Hence

$$I_{n+1} = \frac{2n+1}{2a^2}I_n.$$

Thus for $n = 1, 2, 3, \ldots$ we have

$$I_n = \frac{2n-1}{2a^2}I_{n-1} = \frac{(2n-1)(2n-3)}{(2a^2)^2}I_{n-2} = \cdots = \frac{(2n-1)!!}{(2a^2)^n}I_0.$$

Expressing the double factorial through factorials

$$(2n-1)!! = (2n-1)(2n-3)\cdots 3\cdot 1 = \frac{(2n-1)!}{(2n-2)(2n-4)\cdots 4\cdot 2} = \frac{(2n-1)!}{(n-1)!\,2^{n-1}}$$

and taking into account that

$$I_0 = \int_0^{\infty}\exp(-a^2x^2)\,\mathrm{d}x = \frac{\sqrt{\pi}}{2|a|}, \tag{8}$$

(cf. e.g. A.2(c)) we obtain for $n \geq 1$

$$I_n = \frac{\sqrt{\pi}}{|a|(2a)^{2n}}\frac{(2n-1)!}{(n-1)!}. \tag{9}$$

Inserting (8) and (9) into (7) and taking the term $n = 0$ out of the sum we get

$$\begin{aligned}
\mathrm{FT}\left\{\exp(-a^2x^2)\right\} &= 2A\left[\frac{\sqrt{\pi}}{2|a|}+\frac{\sqrt{\pi}}{|a|}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}\frac{(2n-1)!\,(kX)^{2n}}{(n-1)!\,(2a)^{2n}}\right]= \\
&= A\frac{\sqrt{\pi}}{|a|}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n(kX)^{2n}}{n!\,(2a)^{2n}}\right]= \\
&= A\frac{\sqrt{\pi}}{|a|}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{kX}{2a}\right)^{2n}= \\
&= A\frac{\sqrt{\pi}}{|a|}\sum_{n=0}^{\infty}\frac{1}{n!}\left[-\left(\frac{k}{2a}\right)^2X^2\right]^n.
\end{aligned}$$

The last series is the expansion of the exponential function so that we have the result

$$\mathrm{FT}\left\{\exp(-a^2x^2)\right\}=A\frac{\sqrt{\pi}}{|a|}\exp\left[-\left(\frac{k}{2a}\right)^2X^2\right]. \tag{10}$$

The Fourier transform of the Gaussian function is proportional to the Gaussian function. This fact is often underlined but it is not uniqe. There are many functions which have the same form as their Fourier transform (e.g. $|x|^{-1/2}$ (cf. section 1.3.7), $\sum_{n=-\infty}^{\infty}\delta(x-n)$ (cf. section 4.3) and others).

Note: There are other ways how to calculate the Fourier transform of the Gaussian function and the integral $\int_0^{\infty}\exp\left(-a^2x^2\right)\cos(kxX)\,\mathrm{d}x$, respectively; e.g. [11], p. 731.

In the similar way we can calculate the inverse Fourier transform

$$\begin{aligned}
\mathrm{FT}^{-1}\left\{A\frac{\sqrt{\pi}}{|a|}\exp\left[-\left(\frac{k}{2a}\right)^2X^2\right]\right\} &= AB\,\frac{\sqrt{\pi}}{|a|}\int_{-\infty}^{\infty}\exp\left[-\left(\frac{k}{2a}\right)^2X^2\right]\exp(\mathrm{i}kxX)\,\mathrm{d}X= \\
&= 2AB\,\frac{\sqrt{\pi}}{|a|}\int_0^{\infty}\exp\left[-\left(\frac{k}{2a}\right)^2X^2\right]\cos(kxX)\,\mathrm{d}X= \\
&= 2AB\,\frac{\sqrt{\pi}}{|a|}\int_0^{\infty}\exp\left[-\left(\frac{k}{2a}\right)^2X^2\right]\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(kxX)^{2n}\,\mathrm{d}X= \\
&= 2AB\,\frac{\sqrt{\pi}}{|a|}\sum_{n=0}^{\infty}\frac{(-1)^n(kx)^{2n}}{(2n)!}\int_0^{\infty}X^{2n}\exp\left[-\left(\frac{k}{2a}\right)^2X^2\right]\mathrm{d}X= \\
&= 2AB\,\frac{\sqrt{\pi}}{|a|}\left[\sqrt{\pi}\left|\frac{a}{k}\right|+2\sqrt{\pi}\left|\frac{a}{k}\right|\sum_{n=1}^{\infty}\frac{(-1)^n(kx)^{2n}(2n-1)!\,a^{2n}}{(2n)!\,(n-1)!\,k^{2n}}\right]= \\
&= 2AB\,\frac{\pi}{|k|}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(ax)^{2n}= \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}(-a^2x^2)^n= \\
&= \exp(-a^2x^2).
\end{aligned}$$

The Gaussian function of N variables can be factorized:

$$\exp(-a^2r^2)=\prod_{i=1}^{n}\exp(-a^2x_i^2),\qquad r^2=x_1^2+x_2^2+\cdots+x_N^2.$$

Thus, the Fourier integral can be factorized as well and we get

$$\mathrm{FT}\left\{\exp\left(-a^2r^2\right)\right\}=A^N\left(\frac{\sqrt{\pi}}{|a|}\right)^N\exp\left[-\left(\frac{k}{2a}\right)^2R^2\right],$$

where $R^2=X_1^2+X_2^2+\cdots+X_N^2$. As it was already mentioned this result is important namely in the case $N=2$ for optics of the Gaussian beams.

1.3.4 Example: $f(x) = \exp(-a^2|x|)$

$$\begin{aligned}\mathrm{FT}\left\{\exp(-a^2|x|)\right\} &= A\int_{-\infty}^{\infty}\exp(-a^2|x|)\exp(-\mathrm{i}kxX)\,\mathrm{d}x = \\ &= 2A\int_0^{\infty}\exp\left(-a^2x\right)\cos(kxX)\,\mathrm{d}x\end{aligned}$$

The last integral can be evaluated by two times applied partial integration. Namely, it is

$$\begin{aligned}I &= \int_0^{\infty}\exp\left(-\alpha^2x\right)\cos(\beta x)\,\mathrm{d}x = \frac{\alpha^2}{\beta}\int_0^{\infty}\exp\left(-\alpha^2x\right)\sin(\beta x)\,\mathrm{d}x = \\ &= \frac{\alpha^2}{\beta^2} - \frac{\alpha^4}{\beta^2}\int_0^{\infty}\exp\left(-\alpha^2x\right)\cos(\beta x)\,\mathrm{d}x = \frac{\alpha^2}{\beta^2} - \frac{\alpha^4}{\beta^2}\,I.\end{aligned}$$

Hence

$$I\left(1+\frac{\alpha^4}{\beta^2}\right) = \frac{\alpha^2}{\beta^2},$$

and, consequently,

$$I = \frac{\alpha^2}{\beta^2+\alpha^4}.$$

Thus, the calculated Fourier transform is

$$\mathrm{FT}\left\{\exp\left(-a^2|x|\right)\right\} = A\,\frac{2}{a^2}\,\frac{a^4}{a^4+(kX)^2}. \tag{11}$$

(This rather complicated expression for the result is used as we always try to express the Fourier transform of a real non–negative function in such a way that it is proportional to function which takes the value one at $X = 0$.)

It is not easy to evaluate the inverse Fourier transform of the function (11) as it leads to the Laplace integral:

$$\begin{aligned}\mathrm{FT}^{-1}\left\{A\,\frac{2}{a^2}\,\frac{a^4}{a^4+(kX)^2}\right\} &= BA\,2a^2\int_{-\infty}^{\infty}\frac{\exp(\mathrm{i}kxX)}{a^4+(kX)^2}\,\mathrm{d}X = \\ &= AB\,4a^2\int_0^{\infty}\frac{\cos(kxX)}{a^4+(kX)^2}\,\mathrm{d}X = \\ &= AB\,\frac{4a^2}{|k|}\int_0^{\infty}\frac{\cos(xu)}{a^4+u^2}\,\mathrm{d}u.\end{aligned}$$

The last integral is the Laplace integral (see [11], p. 725, the result provides also [12], 3.723.2)

$$\int_0^{\infty}\frac{\cos\beta t}{\alpha^2+t^2}\,\mathrm{d}t = \frac{\pi}{2\alpha}\,\exp\bigl(-|\alpha\beta|\bigr). \tag{12}$$

Using it we get the expected expression for the calculated inverse Fourier transform

$$\mathrm{FT}^{-1}\left\{A\,\frac{2}{a^2}\,\frac{a^4}{a^4+(kX)^2}\right\} = AB\,\frac{2\pi}{|k|}\,\exp(-a^2|x|) = \exp(-a^2|x|).$$

Note: By the use of the expression (12) for the Laplace integral we get

$$\mathrm{FT}\left\{\frac{a^2}{a^2+x^2}\right\} = \mathrm{FT}\left\{\frac{1}{1+\left(\frac{x}{a}\right)^2}\right\} = A\pi|a|\exp\bigl(-|kaX|\bigr). \tag{13}$$

1.3.5 Example: $f(x) = \mathrm{rect}\left(\frac{x}{2a}\right)$

The function $\mathrm{rect}(x)$ is defined by the relations

$$\mathrm{rect}(x) = \begin{cases} 1, & \text{if } |x| < \frac{1}{2}\,, \\ \frac{1}{2}, & \text{if } |x| = \frac{1}{2}\,, \\ 0, & \text{if } |x| > \frac{1}{2}\,. \end{cases} \tag{14}$$

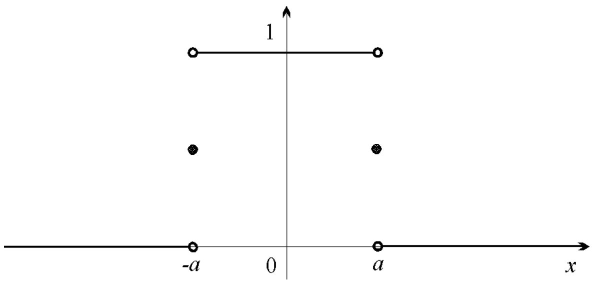

Figure 1: The graph of the function $\mathrm{rect}(\frac{x}{2a})$.

In the theory of diffraction the function is used to specify the transmission function of an infinitely long slit in an opaque screen. If the slit is parallel to the x_2 axis and the x_2 axis passes through its centre, and the width of the slit is $2a$, the transmission function of such a screen is

$$f(x_1, x_2) = \mathrm{rect}\left(\frac{x_1}{2a}\right). \tag{15}$$

Let us calculate the Fourier transform of the function $f(x) = \mathrm{rect}(\frac{x}{2a})$, cf. Figure 1:

$$\begin{aligned} \mathrm{FT}\{\mathrm{rect}\left(\frac{x}{2a}\right)\} &= A\int_{-a}^{a} \exp(-\mathrm{i}kxX)\,\mathrm{d}x = \frac{A}{-\mathrm{i}kX}[\exp(-\mathrm{i}kaX) - \exp(\mathrm{i}kaX)] = \\ &= A2a\,\frac{\sin(kaX)}{kaX}, \end{aligned} \tag{16}$$

cf. Figure 2. It is interesting that the function $\frac{\sin(kaX)}{kaX}$ is not absolutely integrable in the region $(-\infty, \infty)$.

To prove that the function $\frac{\sin z}{z}$ is not absolutely integrable in the region $(-\infty, \infty)$ we divide the region of integration into the intervals $\langle n\pi, (n+1)\pi\rangle$ and estimate the arisen integrals:

$$\int_{-\infty}^{\infty} \left|\frac{\sin z}{z}\right| \mathrm{d}z = 2\int_{0}^{\infty} \frac{|\sin z|}{z}\,\mathrm{d}z = 2\sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{|\sin z|}{z}\,\mathrm{d}z.$$

The first term in the series is the sine–integral

$$\int_0^{\pi} \frac{\sin z}{z}\,\mathrm{d}z = \mathrm{Si}(\pi) > 1.85.$$

All the other terms are larger than $2/[(n+1)\pi]$, because in the integral of each of them $z < (n+1)\pi$ and hence

$$\int_{n\pi}^{(n+1)\pi} \frac{|\sin z|}{z}\,\mathrm{d}z > \frac{1}{(n+1)\pi}\int_0^{\pi} \sin z\,\mathrm{d}z = \frac{2}{(n+1)\pi}.$$

Thus, it is

$$\int_{-\infty}^{\infty} \left|\frac{\sin z}{z}\right| \mathrm{d}z \; > 3,7 + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{n+1} \to \infty,$$

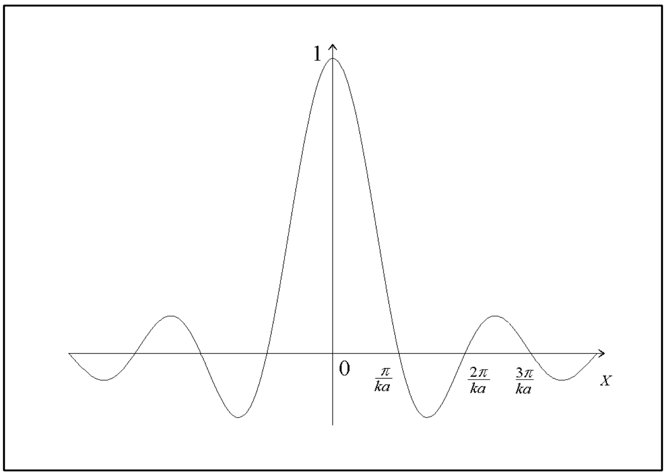

Figure 2: The graph of the function $\frac{\sin(kaX)}{kaX}$.

as the harmonic series $\sum \frac{1}{n}$ is divergent.

Of course, the inverse Fourier transform of the function (16) exists:

$$\begin{aligned}
\mathrm{FT}^{-1}\{\mathrm{FT}\{\mathrm{rect}\left(\frac{x}{2a}\right)\}\} &= BA2a\int_{-\infty}^{\infty}\frac{\sin(kaX)}{kaX}\exp(\mathrm{i}kxX)\,\mathrm{d}X = \\
&= BA2a2\int_{0}^{\infty}\frac{\sin(kaX)}{kaX}\cos(kxX)\,\mathrm{d}X = \\
&= \frac{AB}{k}\,4\int_{0}^{\infty}\frac{\sin(kaX)}{X}\cos(kxX)\,\mathrm{d}X.
\end{aligned}$$

The integral is the Dirichlet discontinuous factor (cf. [11], p. 636, [12], 3.741.2):

$$\int_0^{\infty}\sin(\alpha x)\cos(\beta x)\frac{\mathrm{d}x}{x} = \begin{cases} \frac{\pi}{2}, & \text{if} \quad \alpha > \beta \geq 0, \\ \frac{\pi}{4}, & \text{if} \quad \alpha = \beta > 0, \\ 0, & \text{if} \quad \beta > \alpha \geq 0. \end{cases} \tag{17}$$

Using it we get the expected result

$$\begin{aligned}
\mathrm{FT}^{-1}\{\mathrm{FT}\left\{\mathrm{rect}\left(\frac{x}{2a}\right)\right\}\} &= \frac{AB}{k}\,4\ \mathrm{sgn}(k)\int_0^{\infty}\frac{\sin(|k|aX)}{X}\cos(kxX)\,\mathrm{d}X = \\
&= \begin{cases} AB\,\frac{2\pi}{|k|}\cdot 1 = 1, & \text{if } |x| < a, \\ AB\,\frac{2\pi}{|k|}\cdot\frac{1}{2} = \frac{1}{2}, & \text{if } |x| = a, \\ AB\,\frac{2\pi}{|k|}\cdot 0 = 0, & \text{if } |x| > a. \end{cases} \\
&= \mathrm{rect}\left(\frac{x}{2a}\right).
\end{aligned}$$

If $f(x_1, x_2)$ is the characteristic function of an rectangle,

$$f(x_1, x_2) = \mathrm{rect}\left(\frac{x_1}{2a}\right)\mathrm{rect}\left(\frac{x_2}{2b}\right), \tag{18}$$

the double Fourier integral may be factorize and according to (16) we get

$$F(X_1, X_2) = A^2 4ab\,\frac{\sin(kaX_1)}{kaX_1}\,\frac{\sin(kbX_2)}{kbX_2}. \tag{19}$$

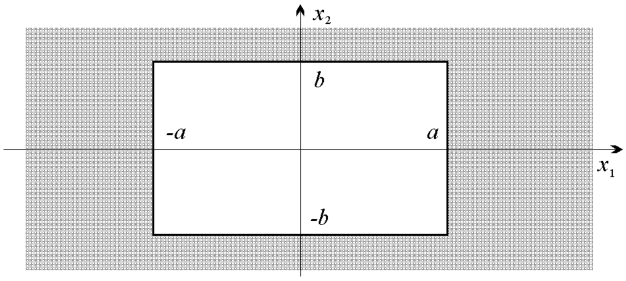

Figure 3: The function $f(x_1, x_2) = \text{rect}\left(\frac{x_1}{2a}\right) \text{rect}\left(\frac{x_2}{2b}\right)$.

Evidently

$$\text{FT}^{-1}\left\{\text{FT}\left\{\text{rect}\left(\frac{x_1}{2a}\right)\text{rect}\left(\frac{x_2}{2b}\right)\right\}\right\} = \begin{cases} 1, & \text{if} \quad |x_1| < a, \ |x_2| < b, \\ \frac{1}{2}, & \text{if} \quad |x_1| < a, \ |x_2| = b \quad \text{or} \quad |x_1| = a, \ |x_2| < b, \\ \frac{1}{4}, & \text{if} \quad |x_1| = a, \ |x_2| = b, \\ 0, & \text{if} \quad |x_1| > a, \ |x_2| > b, \end{cases}$$

in agreement with (18) and with the fundamental theorem.

Let us now calculate the Fourier transform of the characteristic function (15) of infinitely long slit of the width $2a$. With the use of A.2(5) we get

$$\begin{aligned} F(X_1, X_2) &= \text{FT}\left\{\text{rect}\left(\frac{x_1}{2a}\right)\right\} = A^2\, 4a\, \frac{\sin(kaX_1)}{kaX_1} \lim_{b\to\infty} \frac{\sin(kbX_2)}{kX_2} = A^2\, 2a\, \frac{\sin(kaX_1)}{kaX_1}\, 2\pi\delta(kX_2) = \\ &= A^2\, 2a\, \frac{\sin(kaX_1)}{kaX_1}\, \frac{2\pi}{|k|}\, \delta(X_2) = \frac{A}{B}\, 2a\, \frac{\sin(kaX_1)}{kaX_1} \delta(X_2). \end{aligned} \tag{20}$$

Thus, this Fourier transform has the non–zero values only at points of the X_1 axis.

1.3.6 Example: $f(x) = \text{tri}\left(\frac{x}{b}\right)$

The function $\text{tri}(x)$ is defined by relations

$$\text{tri}(x) = \begin{cases} 1 - |x|, & \text{if } |x| \le 1, \\ 0, & \text{if } |x| \ge 1 \end{cases} \tag{21}$$

and the graph of the function $\text{tri}(x/b)$ is in Fig. 4.

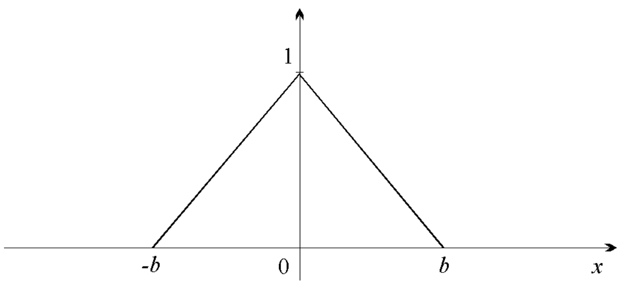

Figure 4: The graph of the function $\text{tri}(x/b)$.

While calculating the Fourier transform we again take advantage of the fact that $\text{tri}(x/b)$ is the even function. The integration by parts provides

$$\begin{aligned} F(X) &= \mathrm{FT}\left\{\mathrm{tri}\left(\frac{x}{b}\right)\right\} = 2A\int_0^b \frac{b-x}{b}\cos(kxX)\,\mathrm{d}x = \frac{2A}{bkX}\int_0^b \sin(kxX)\,\mathrm{d}x = \\ &= -\frac{2A}{b(kX)^2}\cos(kxX)\Big|_0^b = \frac{2A}{b(kX)^2}\left[1-\cos(bkX)\right]. \end{aligned}$$

Expressing the cosine by the sine and cosine of the half of the argument we get the result

$$\mathrm{FT}\left\{\mathrm{tri}\left(\frac{x}{b}\right)\right\} = Ab\left[\frac{\sin(kXb/2)}{kXb/2}\right]^2. \tag{22}$$

It is worth while to notice that

$$\mathrm{FT}\left\{\mathrm{tri}\left(\frac{x}{b}\right)\right\} = \frac{1}{Ab}\left(\mathrm{FT}\left\{\mathrm{rect}\left(\frac{x}{b}\right)\right\}\right)^2. \tag{23}$$

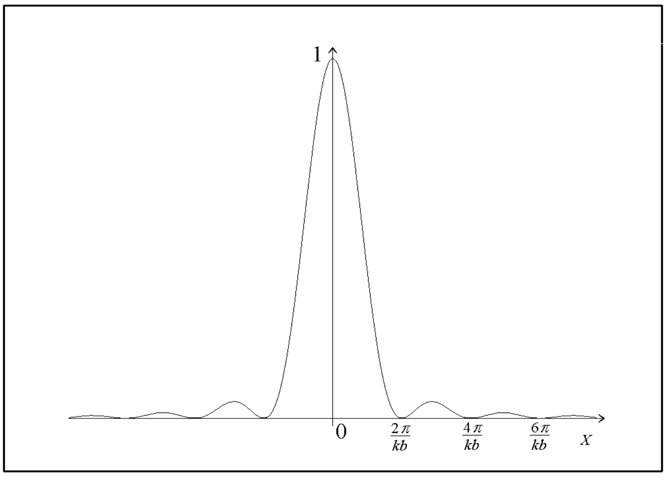

Figure 5: The graph of the function $\left[\frac{\sin(kXb/2)}{kXb/2}\right]^2$.

Let us calculate the inverse Fourier transform

$$\begin{aligned} \mathrm{FT}^{-1}\left\{\mathrm{FT}\left\{\mathrm{tri}\left(\frac{X}{b}\right)\right\}\right\} &= ABb\int_{-\infty}^{\infty}\left[\frac{\sin(kXb/2)}{kXb/2}\right]^2 \exp(\mathrm{i}kxX)\,\mathrm{d}X = \\ &= 2ABb\int_0^{\infty}\left[\frac{\sin(kXb/2)}{kXb/2}\right]^2\cos(kxX)\,\mathrm{d}X = \\ &= AB\frac{4}{|k|}\int_0^{\infty}\left(\frac{\sin u}{u}\right)^2\cos(2xu/b)\,\mathrm{d}u \\ &= \frac{2}{\pi}\int_0^{\infty}\frac{\sin^2 u\cos(2xu/b)}{u^2}\,\mathrm{d}u. \end{aligned}$$

In the Table [12], 3.828.5. we find

$$\begin{aligned} \int_0^{\infty}\frac{\sin^2(ax)\cos(2bx)}{x^2}\,\mathrm{d}x &= \frac{\pi}{2}(a-b),\ \text{if } 0<b<a, \\ &= 0,\ \text{if } b\ge a>0. \end{aligned}$$

Thus,

$$\begin{aligned}\mathrm{FT}^{-1}\left\{\mathrm{FT}\left\{\mathrm{tri}\left(\frac{x}{b}\right)\right\}\right\} &= 1-\left|\frac{x}{b}\right|,\ \text{if } |x|\le b,\\ &= 0,\ \text{if } |x|\ge b.\end{aligned}$$

in agreement with the fundamental theorem.

1.3.7 Example: $f(x)=|x|^{-\frac{1}{2}}$

By substitutions we transform the Fourier integral of this even function into the Fresnel integral in its standard form:

$$\begin{aligned}F(X) &= \mathrm{FT}\left\{|x|^{-\frac{1}{2}}\right\} = 2A\int_0^\infty \frac{\cos kXx}{\sqrt{x}}\,\mathrm{d}x = 4A\int_0^\infty \cos\left(kXt^2\right)\,\mathrm{d}t\ = \\ &= 4A\sqrt{\frac{\pi}{2|kX|}}\int_0^\infty \cos\left(\frac{\pi}{2}v^2\right)\,\mathrm{d}v = 4A\sqrt{\frac{\pi}{2|kX|}}\,\frac{1}{2} = \sqrt{\frac{A}{B}}\,|X|^{-\frac{1}{2}}.\end{aligned}$$

1.4 The Fourier transform and the differentiation

The expressions for the Fourier transform of the derivative and for the derivative of the Fourier transform can be obtained by the differentiation of the definition relations 1.1(1) and 1.1(2).

1.4.1 The Fourier transform and the derivative of function of single variable

By the differentiation of the relation for the inverse Fourier transformation

$$f(x) = B\int_{-\infty}^{\infty} F(X)\exp(\mathrm{i}kXx)\,\mathrm{d}X$$

we get

$$f'(x) = \mathrm{i}kB\int_{-\infty}^{\infty} XF(X)\exp(\mathrm{i}kXx)\,\mathrm{d}X = \mathrm{i}k\,\mathrm{FT}^{-1}\left\{XF(X)\right\}. \tag{1}$$

Similarly, the differentiation of the relation

$$F(X) = A\int_{-\infty}^{\infty} f(x)\exp(-\mathrm{i}kXx)\,\mathrm{d}x$$

leads to the relation

$$F'(X) = -\mathrm{i}kA\int_{-\infty}^{\infty} xf(x)\exp(-\mathrm{i}kXx)\,\mathrm{d}x = -\mathrm{i}k\,\mathrm{FT}\left\{xf(x)\right\}. \tag{2}$$

The Fourier transform of (1) and inverse Fourier transform of (2) provides the formulae

$$\mathrm{FT}\left\{f'(x)\right\} = \mathrm{i}kXF(X), \tag{3}$$

$$\mathrm{FT}^{-1}\left\{F'(X)\right\} = -\mathrm{i}kxf(x). \tag{4}$$

Obviously, by further differentiation we obtain the relations

$$f^{(n)}(x) = (\mathrm{i}k)^n\,\mathrm{FT}^{-1}\left\{X^nF(X)\right\},\quad \text{i.e.}\quad \mathrm{FT}\left\{f^{(n)}(x)\right\} = (\mathrm{i}k)^nX^nF(X), \tag{5}$$

$$F^{(n)}(X) = (-\mathrm{i}k)^n\,\mathrm{FT}\left\{x^nf(x)\right\},\quad \text{i.e.}\quad \mathrm{FT}^{-1}\left\{F^{(n)}(X)\right\} = (-\mathrm{i}k)^nx^nf(x). \tag{6}$$

The relations (1) to (6) are often useful for the calculation of the Fourier transform. We shall illustrate it by a simple and important example.

1.4.2 Example: $f(x) = x/|x| = \operatorname{sgn} x$

It is shown in A.3(13) that the derivative of the function $f(x) = x/|x|$ is

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{|x|}\right) = 2\,\delta(x)$$

and from 1.3(1) it follows that the Fourier transform of that derivative is

$$\mathrm{FT}\left\{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{|x|}\right)\right\} = 2\,\mathrm{FT}\left\{\delta(x)\right\} = 2A.$$

Nevertheless, according to (3) it is also

$$\mathrm{FT}\left\{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{|x|}\right)\right\} = \mathrm{i}kX\,\mathrm{FT}\left\{\frac{x}{|x|}\right\}.$$

By comparison of the right–hand sides of the last two equations we get

$$\mathrm{FT}\left\{\frac{x}{|x|}\right\} = -A\frac{2\mathrm{i}}{kX} = -\frac{\mathrm{i}}{B}\,\frac{\operatorname{sgn} k}{\pi X}. \tag{7}$$

1.4.3 The Fourier transform and the derivative of functions of several variables

We may proceed with functions of several variables in a similar way as with the function of single variable. By the differentiation of the definition relation of the inverse Fourier transform 1.1(2) we get

$$\frac{\partial f(\vec{x})}{\partial x_i} = B^N \int_{-\infty}^{\infty}\!\!\cdots\!\int F(\vec{X})(ikX_i)\exp(\mathrm{i}k\vec{X}\cdot\vec{x})\,\mathrm{d}^N\vec{X}.$$

Hence

$$\frac{\partial f(\vec{x})}{\partial x_i} = \mathrm{i}k\,\mathrm{FT}^{-1}\left\{X_iF(\vec{X})\right\}, \quad \text{tj.} \quad \mathrm{FT}\left\{\frac{\partial f(\vec{x})}{\partial x_i}\right\} = \mathrm{i}kX_iF(\vec{X}). \tag{8}$$

Similarly, the differentiation of the definition relation of the Fourier transform 1.1(1) gives

$$\frac{\partial F(\vec{X})}{\partial X_i} = -\mathrm{i}k\,\mathrm{FT}\left\{x_if(\vec{x})\right\}, \quad \text{i.e.} \quad \mathrm{FT}^{-1}\left\{\frac{\partial F(\vec{X})}{\partial X_i}\right\} = -\mathrm{i}kx_if(\vec{x}). \tag{9}$$

It is more complicated to record the higher order $n = \sum_{i=1}^{N} p_i$ derivatives:

$$\frac{\partial^n f(\vec{x})}{\prod_{i=1}^{N}\partial x_i^{p_i}} = (\mathrm{i}k)^n\,\mathrm{FT}^{-1}\left\{\prod_{i=1}^{N}(X_i^{p_i})\,F(\vec{X})\right\}, \quad \text{i.e.} \quad \mathrm{FT}\left\{\frac{\partial^n f(\vec{x})}{\prod_{i=1}^{N}\partial x_i^{p_i}}\right\} = (\mathrm{i}k)^n\prod_{i=1}^{N}X_i^{p_i}F(\vec{X}), \tag{10}$$

$$\frac{\partial^n F(\vec{X})}{\prod_{i=1}^{N}\partial X_i^{p_i}} = (-\mathrm{i}k)^n\,\mathrm{FT}\left\{\prod_{i=1}^{N}(x_i^{p_i})\,f(\vec{x})\right\}, \quad \text{i.e.} \quad \mathrm{FT}^{-1}\left\{\frac{\partial^n F(\vec{X})}{\prod_{i=1}^{N}\partial X_i^{p_i}}\right\} = (-\mathrm{i}k)^n\prod_{i=1}^{N}x_i^{p_i}f(\vec{x}). \tag{11}$$

It is not usual to define the Fourier transform of vector functions. Nevertheless, if we define the Fourier transform of a vector function $\vec{g}(\vec{x}) = \sum_{e=1}^{n} g_e(\vec{x})\vec{\imath}_e$ as the sum of the Fourier transforms of its Cartesian components $g_e(\vec{x})$, i.e.

$$\mathrm{FT}\left\{\vec{g}(\vec{x})\right\} = \sum_{e=1}^{N}\mathrm{FT}\{g_e(\vec{x})\}\vec{\imath}_e,$$

we may express the Fourier transforms of operations of the vector analysis. For example of the gradient

$$\nabla f(\vec{x}) = \mathrm{i}k\,\mathrm{FT}^{-1}\left\{\vec{X}F(\vec{X})\right\}, \quad \text{i.e.} \quad \mathrm{FT}\{\nabla f(\vec{x})\} = \mathrm{i}k\vec{X}F(\vec{X}), \tag{12}$$

$$\nabla F(\vec{X}) = -\mathrm{i}k\,\mathrm{FT}\{\vec{x}f(\vec{x})\}, \quad \text{i.e.} \quad \mathrm{FT}^{-1}\left\{\nabla F(\vec{X})\right\} = -\mathrm{i}k\vec{x}f(\vec{x}), \tag{13}$$

or of the application of Laplacian ∇^2

$$\nabla^2 f(\vec{x}) = -k^2\,\mathrm{FT}^{-1}\left\{X^2F(\vec{X})\right\}, \quad \text{i.e.} \quad \mathrm{FT}\left\{\nabla^2 f(\vec{x})\right\} = -k^2X^2F(\vec{X}), \tag{14}$$

$$\nabla^2 F(\vec{X}) = -k^2\,\mathrm{FT}\left\{x^2f(\vec{x})\right\}, \quad \text{i.e.} \quad \mathrm{FT}^{-1}\left\{\nabla^2F(\vec{X})\right\} = -k^2x^2f(\vec{x}). \tag{15}$$

In the literature it is often investigated the Fourier transform of the Taylor or the Maclaurin series and of the moments related with the series (cf. e.g. [13], pp. 18, 47). We shall not do it as it is rather aside of the topics of the present course.

1.5 The Fourier transform of the Fourier transform

For some applications — e.g. for the description of image formation by a telescopic system — it may be useful to know the result of two times applied Fourier transform. Therefore we will calculate $\mathrm{FT}\{\mathrm{FT}\{f(\vec{x})\}\}$ and $\mathrm{FT}^{-1}\left\{\mathrm{FT}^{-1}\{f(\vec{x})\}\right\}$. We shall start with the definition of the Fourier transform, by the interchange of the order of integrations we get the expression A.2(5) of the Dirac distribution, and using its sifting property A.1(2) we get the result:

$$\begin{aligned}
FT\{\mathrm{FT}\{f(\vec{x})\}\} &= A^N \underset{\vec{X}'}{\int\cdots\int} A^N \underset{\vec{x}'}{\int\cdots\int} f(\vec{x}')\exp\left(-\mathrm{i}k\vec{X}'\cdot\vec{x}'\right)\mathrm{d}^N\vec{x}'\exp\left(-\mathrm{i}k\vec{X}'\cdot\vec{x}\right)\mathrm{d}^N\vec{X}' = \\
&= A^{2N} \underset{\vec{x}'}{\int\cdots\int} f(\vec{x}') \underset{\vec{X}'}{\int\cdots\int} \exp\left[-\mathrm{i}k\vec{X}'(\vec{x}'+\vec{x})\right]\mathrm{d}^N\vec{X}'\,\mathrm{d}^N\vec{x}' = \\
&= A^{2N}\left(\frac{2\pi}{|k|}\right)^N \underset{\vec{x}'}{\int\cdots\int} f(\vec{x}')\;\delta(\vec{x}'+\vec{x})\,\mathrm{d}^N\vec{x}' = \\
&= \left(\frac{A}{B}\right)^N f(-\vec{x}).
\end{aligned}$$

Similarly

$$\begin{aligned}
FT^{-1}\left\{\mathrm{FT}^{-1}\{f(\vec{x})\}\right\} &= B^N \underset{\vec{X}'}{\int\cdots\int} B^N \underset{\vec{x}'}{\int\cdots\int} f(\vec{x}')\exp\left(\mathrm{i}k\vec{X}'\cdot\vec{x}'\right)\mathrm{d}^N\vec{x}'\exp\left(\mathrm{i}k\vec{X}'\cdot\vec{x}\right)\mathrm{d}^N\vec{X}' = \\
&= B^{2N} \underset{\vec{x}'}{\int\cdots\int} f(\vec{x}') \underset{\vec{X}'}{\int\cdots\int} \exp\left[\mathrm{i}k\vec{X}'(\vec{x}'+\vec{x})\right]\mathrm{d}^N\vec{X}'\,\mathrm{d}^N\vec{x}' = \\
&= B^{2N}\left(\frac{2\pi}{|k|}\right)^N \underset{\vec{x}'}{\int\cdots\int} f(\vec{x}')\;\delta(\vec{x}'+\vec{x})\,\mathrm{d}^N\vec{x}' = \\
&= \left(\frac{B}{A}\right)^N f(-\vec{x}).
\end{aligned}$$

These results

$$\mathrm{FT}\{\mathrm{FT}\{f(\vec{x})\}\} = \left(\frac{A}{B}\right)^N f(-\vec{x}), \tag{1}$$

$$\mathrm{FT}^{-1}\left\{\mathrm{FT}^{-1}\{f(\vec{x})\}\right\} = \left(\frac{B}{A}\right)^N f(-\vec{x}), \tag{2}$$

can be rewriten into the form

$$\frac{1}{A^N}\,\mathrm{FT}\,\{f(\vec{x})\} \;=\; \frac{1}{B^N}\,\mathrm{FT}^{-1}\,\{f(-\vec{x})\}\,,. \tag{3}$$

The validity of the relation (3) can be obtained also immediately from the definition of the Fourier transform (e.g. by replacing $\vec{x}$ by $-\vec{x}$ in the integral 1.1(1)).

1.6 The relation of $\mathrm{FT}\,\{f(\vec{x})\}$ and $\mathrm{FT}\,\{f^*(\vec{x})\}$ and the relation of $\mathrm{FT}^{-1}\left\{F(\vec{X})\right\}$ and $\mathrm{FT}^{-1}\left\{F^*(\vec{X})\right\}$

In optics the product $f(\vec{x})f^*(\vec{x})$ has often the meaning of the intensity distribution in the image plane and in the Fourier optics the Fourier transform of that product is calculated. As we shall see in Sect. 7.6, in this connection it is necessary to know how the Fourier transform of a function $f(\vec{x})$ and the Fourier transform of the complex conjugate function $f^*(\vec{x})$ are related. Similarly, in structure analysis it is calculated the inverse Fourier transform of the intensity distribution in diffraction patterns, that is of the product $F(\vec{X})F^*(\vec{X})$. For that it is necessary to know what is the inverse Fourier transform of the function $F^*(\vec{X})$.

Evidently, in general $\mathrm{FT}\,\{f^*(\vec{x})\} \neq F^*(\vec{X})$. Therefore, let us calculate $\mathrm{FT}\,\{f^*(\vec{x})\}$:

$$\begin{aligned}\mathrm{FT}\,\{f^*(\vec{x})\} &= A^N \int_{-\infty}^{\infty}\!\!\cdots\!\int f^*(\vec{x})\exp\bigl(-\mathrm{i}k\vec{X}\cdot\vec{x}\bigr)\,\mathrm{d}^N\vec{x} = A^N \int_{-\infty}^{\infty}\!\!\cdots\!\int f^*(\vec{x})\exp\bigl[\mathrm{i}k(-\vec{X})\cdot\vec{x}\bigr]\,\mathrm{d}^N\vec{x} = \\ &= F^*(-\vec{X}).\end{aligned}$$

Hence,

$$\mathrm{FT}\,\{f^*(\vec{x})\} = F^*(-\vec{X}). \tag{1}$$

Similarly,

$$\begin{aligned}FT^{-1}\left\{F^*(\vec{X})\right\} &= B^N \int_{-\infty}^{\infty}\!\!\cdots\!\int F^*(\vec{X})\exp\bigl(\mathrm{i}k\vec{X}\cdot\vec{x}\bigr)\,\mathrm{d}^N\vec{X} = \\ &= B^N \int_{-\infty}^{\infty}\!\!\cdots\!\int F^*(\vec{X})\exp\bigl[-\mathrm{i}k\vec{X}\cdot(-\vec{x})\bigr]\,\mathrm{d}^N\vec{X} = f^*(-\vec{x}).\end{aligned}$$

So that

$$FT^{-1}\left\{F^*(\vec{X})\right\} = f^*(-\vec{x}). \tag{2}$$

* * *

We could continue in listing properties of the Fourier transform in the style used up to now, i.e. in such a way that we would derive or merely present coresponding formulae. We will however change the manner of presentation now and we shall first discuss the meanings of the Fourier transform in the diffraction theory (chapter 2) and only then we come back to the properties of the Fourier transform. Each property we will then not only specify mathematically, but also illustrate by diffraction patterns.

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